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TOTAL PROGENY IN AGE–DEPENDENT BRANCHING PROCESSES WITH STATE–DEPENDENT IMMIGRATION *

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We study critical Bellman-Harris branching processes with immigration in the state zero. For these processes the limiting behavior of the total number of particles in the interval $[0,t]$ is investigated.

Definition and basic result. A model of branching process with state-dependent immigration was first investigated by Foster (1971) and Pakes (1971a). They considered a modification of the Galton-Watson branching process which admits immigration only in the state zero. Later this model was generalized for Markov branching processes by Yamazato (1977) and for Bellman-Harris processes by Mitov and Yanev (1985), (1989). An interesting characteristic of the processes is the total number of particles born in the time interval $[0,t]$. It has been studied for different models of branching processes with and without immigration (see e.g. Pakes(1971b), (1972), Kulkarni and Pakes (1983) and references therein).

The aim of this note is to extend the results for Galton-Watson processes with state-dependent immigration obtained in Kulkarni and Pakes (1983) to the Bellman-Harris processes with state-dependent immigration (BHIO).

The definition of BHIO is given in Mitov and Yanev (1985) as follows. Let on the probability space $(\Omega, \mathcal{A}, P)$ be given:

i) A set $X = \{X_i, i = 1, 2, ...\}$ of independent, identicaly distributed (i.i.d.), positive random variables (r.v.) with a common cumulative distribution function (c.d.f.) $K(x) = P(X_i \leq x),\ K(0) = 0$

ii) An independent of X set $Z = \{Z_i(t), t \geq 0, i = 1, 2, ...\}$ of independent Bellman-Harris branching processes with probability generating function (p.g.f.) of initial number of particles $f(s), |s| \leq 1,\ f(0) = 0$, particle-life c.d.f. $G(t),\ G(0) = 0$ and offspring p.g.f. $h(s), |s| \leq 1$.

Denote by T_i the life-period of the process $\{Z_i(t)\}$, i.e., T_i is a r.v. such that

$$Z_i(t) > 0, 0 \leq t < T_i, \quad Z_i(T_i) = 0.$$

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The sequence $U_i = X_i + T_i, \quad i = 1, 2, \ldots$ defines the renewal process

$$S_0 = 0, S_{n+1} = S_n + U_{n+1}, \ n \geq 1, \quad N(t) = \max\{n : S_n \leq t\}.$$

Now, a Bellman-Harris process with state dependent immigration $Z(t)$ is defined as follows:

$$Z(t) = \begin{cases} Z(t - S_{N(t)} - X_{N(t)+1}) & t - S_{N(t)} - X_{N(t)+1} \geq 0 \\ 0, & t - S_{N(t)} - X_{N(t)+1} < 0 \end{cases}$$

By the definition it is clear that the process $\{Z(t)\}$ is regenerative with embedded renewal process $\{S_n\}$.

The total number of particles in the process $Z(t)$ born in the interval $[0, t]$ is defined by the integral

$$W(t) = \int_0^t Z(x)dx.$$

From now on we assume that the following conditions hold:

i) $h'(1) = 1, \quad 0 < h''(1) = \sigma^2 < \infty$, (the critical case).

ii) $G(t)$ is non-lattice, $\mu = \int_0^\infty t dG(t) < \infty$, and $1 - G(t) = o(1/t^2)$ as $t \to \infty$.

iii) $K(t)$ is non-latice $\nu = \int_0^\infty t dK(t) < \infty$.

iv) $\beta = f'(1) = EZ_i(0) < \infty$.

The basic result is given in the following theorem.

Theorem 1. *Let i)-iv) hold. Then as* $t \to \infty$

$$(\log t/t)^2 W(t) \ \stackrel{d}{\to} \ \frac{\sigma^2}{4\mu^2} W,$$

where W *is a stable random variable with index* $1/2$.

Proof. a) By the regenerative properties of the process $\{Z(t)\}$ it is clear that almost surely

$$\sum_{i=1}^{N(t)} W_i \leq W(t) \leq \sum_{i=1}^{N(t)+1} W_i, \tag{1}$$

where

$$W_i = \int_0^{T_i} Z_i(t)dt,$$

is the total number of particles in the process $Z_i(t)$ during its life-period.

b) It has been noted in Jagers (1975) that the random variable W_i has the same distribution as the total number of particles in a simple Galton-Watson branching process having the same p.g.f. of ancestors $f(s)$ and the same offspring p.g.f. $h(s)$. Appealing to this observation we can conclude that under the condition *i)* (see Kulkarni and Pakes

(1983))

(2) $$n^{-2}\sum_{i=1}^{n} W_i \ \stackrel{d}{\to} (\beta/\sigma)^2 W,\ t\to\infty.$$

c) Under the conditions $i)$-$iv)$ it is known that (see Mitov and Yanev (1985))

(3) $$P(U_i > t) \sim \gamma t^{-1},\ \ t\to\infty.$$

where $\gamma = 2\beta\mu/\sigma^2$.

d) By (3) and the weak law of large numbers (see Feller (1984), vol.2., Th.2,Sect. VII.8), it follows that

(4) $$S_n/b(\gamma n) \stackrel{P}{\to} 1,\ \ n\to\infty,$$

where $b(x)$ is the inverse function of $x/\log x$ for $x \geq 1$. Hence, from $P(N(t) \geq n) = P(S_n \leq t)$ and (4) and the fact that $b(x)$ is a regularly varying function with exponent 1, it is not difficult to prove that

(5) $$\frac{\gamma \log t}{t} N(t) \ \stackrel{P}{\to}\ 1,\ \ t\to\infty.$$

e) Now, we are in a position to prove that

(6) $$(\gamma\log t/t)^2 \sum_{i=1}^{N(t)} W_i \ \stackrel{d}{\to}\ (\beta/\sigma)^2 W.$$

Denote by $C(x)$ the c.d.f. of $(\beta/\sigma)^2 W$.

Let $\varepsilon > 0$ be fixed. There exists $T > 0$ such that for $t \geq T$

(7) $$P\Big(|\frac{\gamma\log t}{t} N(t) - 1| > \varepsilon\Big) < \varepsilon.$$

Hence,

(8) $$\begin{aligned} &P\Big(\Big(\frac{\gamma\log t}{t}\Big)^2 \sum_{i=1}^{N(t)} W_i \leq x\Big) \\ &= P\Big(\Big(\frac{\gamma\log t}{t}\Big)^2 \sum_{i=1}^{N(t)} W_i \leq x, \frac{\gamma\log t}{t} N(t) \in (1-\varepsilon, 1+\varepsilon)\Big) \\ &+ P\Big(\Big(\frac{\gamma\log t}{t}\Big)^2 \sum_{i=1}^{N(t)} W_i \leq x,\ \frac{\gamma\log t}{t} N(t) \bar{\in} (1-\varepsilon, 1+\varepsilon)\Big) \\ &= P_1(t) + P_2(t). \end{aligned}$$

From (7) we obtain for t sufficiently large that

$$P_1(t) = P\Big(\Big(\frac{\gamma \log t}{t}\Big)^2 \sum_{i=1}^{N(t)} W_i \le x \,|\, \frac{\gamma \log t}{t} N(t) \in (1-\varepsilon, 1+\varepsilon)\Big)$$

$$\times P\Big(\frac{\gamma \log t}{t} N(t) \in (1-\varepsilon, 1+\varepsilon)\Big)$$

$$\le P\Big(\Big(\frac{\gamma \log t}{t}\Big)^2 \sum_{i=1}^{[\frac{\gamma \log t}{t}(1-\varepsilon)]} W_i \le x\Big),$$

and similarly

$$P_1(t) \ge P\Big(\Big(\frac{\gamma \log t}{t}\Big)^2 \sum_{i=1}^{[\frac{\gamma \log t}{t}(1+\varepsilon)]+1} W_i \le x\Big)(1-\varepsilon).$$

Now, by (2) and the fact that $\Big(\frac{\gamma \log t}{t}\Big)^2$ is a regularly varying function with exponent -2 we see that

(9) $$\limsup_{t\to\infty} P_1(t) \le C\Big(\frac{x}{(1-\varepsilon)^2}\Big) \qquad \liminf_{t\to\infty} P_1(t) \ge C\Big(\frac{x}{(1+\varepsilon)^2}\Big)(1-\varepsilon).$$

On the other hand, from (7) we obtain for $P_2(t)$

(10) $$P_2(t) = P\Big(\Big(\frac{\gamma \log t}{t}\Big)^2 \sum_{i=1}^{N(t)} W_i \le x \,|\, \frac{\gamma \log t}{t} N(t) \bar{\in} (1-\varepsilon, 1+\varepsilon)\Big)$$
$$\times P\Big(\frac{\gamma \log t}{t} N(t) \bar{\in} (1-\varepsilon, 1+\varepsilon)\Big) < \varepsilon.$$

Finally, (8), (9) and (10) prove (6) because ε was arbitrary.

Taking into account that $\gamma = 2\beta\mu/\sigma^2$ and (6) one can see that as $t \to \infty$,

$$(\log t/t)^2 \sum_{i=1}^{N(t)} W_i \quad \xrightarrow{d} \quad \frac{\beta^2}{\sigma^2}\frac{1}{\gamma^2} W = \frac{\sigma^2}{4\mu^2} W.$$

The last relation and (1) complete the proof of the theorem.

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ОБЩ БРОЙ НА ЧАСТИЦИТЕ В РАЗКЛОНЯВАЩ СЕ ПРОЦЕС СЪС ЗАВИСЕЩА ОТ СЪСТОЯНИЕТО ИМИГРАЦИЯ

Косто Вълов Митов

Разглеждат се критически разклоняващи се процеси на Белман-Харис с имиграция в нулата. За тези процеси се изследва граничното поведение на общият брой частици, родени в интервала $[0, t]$.