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MAXIMUM FAMILY SIZE IN BRANCHING PROCESSES WITH STATE–DEPENDENT IMMIGRATION

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The number W_n of offspring of the most prolific particle in the n-th generation of a simple branching process with state-dependent immigration is studied. Limit theorems for W_n and EW_n are proved. The results are obtained by combining the methods of [8] with known behavior of the population size in branching processes with state–dependent immigration.

1. Introduction. Let on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ be given:

i) A set $X = \{X_i(n), i, n = 1, 2, \ldots\}$ of independent identically distributed (i.i.d.) non-negative integer valued random variables with common probability generating function (p.g.f.) $f(s)$ and cumulative probability distribution function (c.d.f.) $F(x)$.

ii) An independent of X set $Y = \{Y_n,\ n = 0, 1, 2, \ldots\}$ of i.i.d. positive integer valued random variables with common p.g.f. $g(s)$.

We consider a branching process with immigration in the state zero only, defined by

$$Z_0 = Y_0\ ,\quad Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n) + I_{\{Z_{n-1}=0\}} Y_n,\quad n = 1, 2, \ldots$$

where I_A stands for the indicator of A, and $\sum_{i=1}^{0} \cdot = 0$.

Usually, Z_n is interpreted as the number of particles living in the nth generation of an evolving population. The process starts with a positive random number of ancestors Y_0 at time $n = 0$ and evolves as a Bienaymé-Galton-Watson (BGW) branching process up to its visit to state zero. At the next moment, $Y_1 > 0$ new particles immigrate and the process starts again, and so on. This model of branching process was introduced by Foster [3] and Pakes [5] and was investigated in several papers later. In [5] one can find another interpretation of the process $\{Z_n\}$ in terms of queueing theory.

In the present work we focus our attention on the sequence $\{W_n\}_{n=0}^{\infty}$ defined by

$$W_0 = Y_0,\quad W_n = \begin{cases} \max\limits_{1\le i\le Z_{n-1}} \{X_i(n)\}, & Z_{n-1} > 0 \\ 0, & Z_{n-1} = 0 \end{cases},\quad n = 1, 2, \ldots \tag{1}$$

The random variable W_n can be interpreted as the number of 'children' of the most prolific among the particles living in the $(n-1)$st generation. There have been several

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recent works developing results for certain kinds of extremes in branching processes, see e.g. [2] and references therein. The results obtained here are closely related to those in [8], where the sequence W_n is studied when $\{Z_n\}$ is a BGW process without immigration. Note that in [6] the case of critical reproduction process is investigated by means of the regenerative properties of $\{Z_n\}$. In the present work the subcritical, critical, and supercritical cases are all studied applying the methods of [8].

Let $\Phi_n(s)$ be p.g.f. of Z_n. It is not difficult to check that (1) is equivalent to

$$\mathrm{P}\{W_n \le x\} = \sum_{k=0}^{\infty} \mathrm{P}\{Z_{n-1} = k\} F^k(x) = \Phi_{n-1}(F(x)). \tag{2}$$

Relation (2) is the basis of obtaining limit theorems for W_n and $\mathrm{E}W_n$ which are presented in sections 3 through 5. Next section contains some preliminary results.

2. Preliminaries. Let us denote $m = \mathrm{E}X_i(n)$ and $\mu = \mathrm{E}Y_n$. We shall treat the subcritical ($m < 1$), critical ($m = 1$), and supercritical ($1 < m < \infty$) cases separately.

It is known (see [5]) that the p.g.f. $\Phi_n(s)$ of Z_n satisfies the equation

$$\Phi_n(s) = \Phi_{n-1}(f(s)) - (1 - g(s))\Phi_{n-1}(0),$$

and

$$EZ_n = \mu m^n + \mu \sum_{i=0}^{n-1} m^i \Phi_{n-1-i}(0). \tag{3}$$

Further on, we will need the following results proved in [8]. Recall that a nondegenerate c.d.f. $H(x)$ is max-stable if and only if for a c.d.f. $F(x)$ there exist functions $a(n) > 0$ and $b(n)$ such that

$$\lim_{n\to\infty} F(a(n)x + b(n)) = H(x), \tag{4}$$

weakly. If (4) holds then $F(x)$ is said to belong to the domain of attraction of $H(x)$, i.e., $F \in D(H)$. According to the classical Gnedenko's result, $H(x) = \exp\{-h(x)\}$, where $h(x)$ is of the type of one of the following three classes:

$$\begin{array}{llll} h(x) = (-x)^a & \text{for} \quad x \in (-\infty, 0), & = 1, & x \in [0, \infty) \\ h(x) = x^{-a} & \text{for} \quad x \in (0, \infty), & = 0, & x \in (-\infty, 0] \\ h(x) = e^{-x} & \text{for} \quad x \in (-\infty, \infty), & & \end{array}$$

where $a > 0$. Moreover, $F \in D(\exp\{-x^{-a}\}),\ a > 0$ if and only if for $x > 0$

$$1 - F(x) = x^{-a} L(x)\ , \tag{5}$$

where $L(\cdot)$ is a slowly varying at infinity function (s.v.f.), see e.g. [9], Prop.1.11.

Lemma 1. ([8]) *Let $\{\eta_i(n), i = 1, 2, \ldots\}$ be sequences of independent for each $n = 0, 1, 2, \ldots$ random variables with common c.d.f. $F(x)$ for which (4) holds. Let $\{\nu(n), n = 0, 1, 2, \ldots\}$ be a sequence of nonnegative integer valued random variables independent of $\{\eta_i(n)\}$ for each $n = 0, 1, 2, \ldots$*

If there exist a function $r : \mathrm{N} \to \mathrm{R}$ with $r(n) \to \infty$ as $n \to \infty$, and a random variable ν such that

$$\lim_{n\to\infty} \mathrm{P}\{\frac{\nu(n)}{r(n)} \le x | B_n\} = \mathrm{P}\{\nu \le x\},$$

weakly, where B_n is a sequence of events such that $\{\nu(n) \geq 0\} \subset B_n$, then

$$\lim_{n\to\infty} \mathrm{P}\Big\{\frac{\max\limits_{1\leq i\leq \nu(n)}\{\eta_i(n)\} - b(r(n))}{a(r(n))} \leq x|B_n\Big\} = \chi(h(x)),$$

where $a(n)$ and $b(n)$ are defined in (4), and $\chi(u)$ is the Laplace transform of ν.

Lemma 2. ([8]) *Let V_n be a sequence of nonnegative random variables such that $\lim_{n\to\infty} \mathrm{P}\{V_n \leq x|B_n\} = \mathrm{P}\{V \leq x\}$ weakly, for a sequence of events B_n, and let $\mathrm{E}V < \infty$. If*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} \mathrm{P}\{V_n > j|B_n\} = 0,$$

then

$$\lim_{n\to\infty} \mathrm{E}V_n = \mathrm{E}V.$$

3. Subcritical case. Suppose that

$$0 < m < 1, \quad \mathrm{E}(\log Y_n) < \infty. \tag{6}$$

Pakes in [5] proves that (6) are necessary and sufficient conditions for the existence of a limiting stationary distribution for Z_n

$$\pi_j = \lim_{n\to\infty} \mathrm{P}\{Z_n = j\}, \quad j = 0, 1, 2, \ldots,$$

such that its p.g.f.

$$\Pi(s) = \sum_{j=0}^{\infty} \pi_j s^j = 1 - \pi_0 \sum_{n=0}^{\infty} (1 - g(f_n(s))),$$

where $f_n(s)$ is the nth functional iterate of $f(s)$, i.e., $f_0(s) = s, f_n(s) = f(f_{n-1}(s))$ and

$$\pi_0 = \{1 + \sum_{n=0}^{\infty} (1 - g(f_n(0)))\}^{-1} \in (0, 1).$$

Moreover, see [5],

$$\lim_{n\to\infty} \mathrm{E}Z_n = \Pi'(1-) = \pi_0 \mu/(1 - m). \tag{7}$$

Now, we are in a position to prove the following result.

Theorem 1. *If (6) hold, then*

(i) $\lim\limits_{n\to\infty} \mathrm{P}\{W_n \leq x\} = \Pi(F(x)), \quad x \in \mathrm{R},$

(ii) $\lim\limits_{n\to\infty} \mathrm{E}W_n = \sum\limits_{k=0}^{\infty} (1 - \Pi(F(k))).$

Proof. (i) The assertion follows from (2) appealing to the continuity theorem for p.g.f.'s, i.e., $\lim_{n\to\infty} \Phi_n(s) = \Pi(s)$.

(ii) We shall apply Lemma 2. First, for $j \geq 1$

$$P\{W_n > j\} = \sum_{k=1}^{\infty} P\{\max_{1\leq i\leq Z_{n-1}} \{X_i(n)\} > j | Z_{n-1} = k\} P\{Z_{n-1} = k\}$$

(8)
$$= \sum_{k=1}^{\infty} (1 - F^k(j)) P\{Z_{n-1} = k\} \leq \sum_{k=1}^{\infty} (1 - F(j)) k P\{Z_{n-1} = k\}$$

$$= (1 - F(j)) EZ_{n-1}.$$

Now, (7) and (8) imply

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P\{W_n > j\} \leq \lim_{N\to\infty} \frac{\pi_0 \mu}{1-m} \sum_{j=N+1}^{\infty} (1 - F(j)) = 0$$

Thus, by Lemma 2

$$\lim_{n\to\infty} EW_n = \int_0^{\infty} 1 - \Pi(F(x)) dx,$$

which is equivalent to (ii). The proof of the theorem is completed.

Notice that, Theorem 1 and (7) imply

(9)
$$\lim_{n\to\infty} EW_n \leq m \lim_{n\to\infty} EZ_n = \pi_0 \mu \frac{m}{1-m}.$$

Example 1. *Let $f(s) = (1 + m - ms)^{-1}$ and $g(s) = 1 - (\mu/m) \log(1 + m - ms)$, where $0 < m < 1$. It is not difficult to check that $EX_i(n) = m$ and $EY_n = \mu$. In this case $\Pi(s)$ has the closed form $\Pi(s) = (m - \mu \log(1 - ms))/(m - \mu \log(1 - m))$. Now, by Theorem 1*

$$\lim_{n\to\infty} P\{W_n \leq x\} = \frac{m - \mu \log(1 - mF(x))}{m - \mu \log(1 - m)} \ ,$$

and

$$\lim_{n\to\infty} EW_n = \mu \ \frac{m + \sum_{k=0}^{\infty} \log\{(1 - mF(x))/(1 - m)\}}{m - \mu \log(1 - m)},$$

where for integer k we have $F(k) = (1 + m)^{k+1} - m^{k+1}$.

Using (9) one can obtain

$$\lim_{n\to\infty} W_n \leq \frac{\mu m}{m - \mu \log(1 - m)} \ \frac{m}{1-m} \ .$$

It is worth comparing this upper bound for W_n with that in [8], Example 4.1, where if $\{Z_n^*\}$ is the ordinary BGW process with the same offspring p.g.f. $f(s)$

$$\lim_{n\to\infty} E(W_n | Z_n^* > 0) \leq \frac{m}{1-m}.$$

4. Critical case. Assume that

(10)
$$m = 1, \quad EX_i^2(n) < \infty, \quad \text{and} \quad 0 < \mu < \infty.$$

It is proved by Foster [3] that under (10)

$$\lim_{n\to\infty} P\Big(\frac{\log Z_n}{\log n} \le x\Big) = x \ , \tag{11}$$

for $0 < x < 1$. We will prove the following result for W_n.

Theorem 2. *Let (10) hold and*

$$\lim_{n\to\infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n+1)} = 1 \ . \tag{12}$$

Then for $0 < x < 1$,

$$\lim_{n\to\infty} P\Big(\frac{\log U(W_n)}{\log n} \le x\Big) = x, \tag{13}$$

where $U(y) = 1/(1 - F(y))$.

Proof. It follows from (12) that (cf. [5], p.24) there exists a sequence $\{u_n\}$ such that for $y > 0$,

$$\lim_{n\to\infty} \frac{1 - F(u_n)}{n^{-x}} = y \ . \tag{14}$$

On the other hand, (11) implies (see [3]) for $0 < x < 1$

$$\lim_{n\to\infty} E(\exp\{-uZ_n n^{-x}\}) = \lim_{n\to\infty} f_n(\exp\{-un^{-x}\}) = x \ .$$

Since $\lim_{n\to\infty}(1 - F(u_n)) = 0$, we have as $n \to \infty$,

$$\begin{aligned} P(W_n > u_n) &= 1 - f_{n-1}(F(u_n)) = 1 - f_{n-1}(\exp\{\ln F(u_n)\}) \\ &= 1 - f_{n-1}(\exp\{-(1 - F(u_n))(1 + o(1))\}) = 1 - f_{n-1}(\exp\{-yn^{-x}(1 + o(1))\}) \\ &\to 1 - x \ . \end{aligned} \tag{15}$$

Further, from (14), using Lemma 2.2.1 in [4], one can obtain for $0 < x < 1$,

$$\begin{aligned} &\lim\nolimits_{n\to\infty} P\Big(\frac{1 - F(W_n)}{n^{-x}} \le y\Big) \\ &= \lim\nolimits_{n\to\infty} P\Big(\frac{1 - F(W_n)}{n^{-x}} \le \frac{1 - F(u_n)}{n^{-x}} + y - \frac{1 - F(u_n)}{n^{-x}}\Big) \\ &= \lim\nolimits_{n\to\infty} P(W_n > u_n) \ . \end{aligned}$$

From here, taking into account (15), we obtain for $U(z) = 1/(1 - F(z))$,

$$\lim_{n\to\infty} P(U(W_n) \le n^x/y) = x \ ,$$

which with $y = 1$ implies (13). The proof is completed.

Remark. It is known (cf. [4], Cor.2.4.1) that (12) is a necessary condition for a c.d.f. $F(x)$ to belong to the domain of attraction of one of the three distribution lows given in Section 1. In particular if $F(x)$ satisfies (5) then it also satisfies (12). On the other hand, one can verify that (12) is not true for geometric and Poisson distributions (see also [8], Remark 2).

5. Supercritical case. Suppose that

$$1 < m < \infty, \quad 0 < \mu < \infty. \tag{16}$$

It is known (see [5]) that under the conditions (16) there exists a sequence $\{C_n\}$ with

$C_n \to \infty$ and $C_{n+1}/C_n \to m$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \frac{Z_n}{C_n} = Z,$$

almost surely, where Z is a continuous positive random variable with Laplace transform $\psi(u)$ given by

$$\psi(u) = g(\varphi(u)) - \sum_{n=0}^{\infty}(1 - f(\varphi(\frac{u}{m^n})))\Phi_n(0), \tag{17}$$

and $\varphi(\cdot)$ satisfies the equation $\varphi(mu) = f(\varphi(u))$. If in addition

$$\mathrm{E}X_i(n)\log(1 + X_i(n)) < \infty, \tag{18}$$

then C_n can be chosen as m^n.

Recall that the de Bruijn conjugate of a s.v.f. $L(x)$ is a s.v.f. $L^*(x)$, unique up to asymptotic equivalence, with (see Seneta (1976), Thm. 1.5)

$$\lim_{x\to\infty} L(x)L^*(xL(x)) = 1, \quad \lim_{x\to\infty} L^*(x)L(xL^*(x)) = 1.$$

We shall prove the following theorem.

Theorem 3. *Let (16) hold and $\psi(u)$ be the Laplace transform in (17).*

(i) If (4) holds, then for any $x \in \mathrm{R}$,

$$\lim_{n\to\infty} \mathrm{P}\Big\{\frac{W_n - b(C_n)}{a(C_n)} \le x\Big\} = \psi(h(x)).$$

(ii) If (5) and (18) hold, then

$$\lim_{n\to\infty} \frac{\mathrm{E}W_n}{m^{n/a}(L_1(m^{n/a}))^{1/a}} = \int_0^\infty (1 - \psi(x^{-a}))dx,$$

where $L_1(\cdot)$ is the de Bruijn conjugate of $1/L(\cdot)$, with

$$\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1, \quad \lim_{x\to\infty} L_1(x)/L(xL_1(x)) = 1.$$

Proof. (i) Since $\{Z_n/C_n\}$ converges to Z almost surely, and hence in distribution, the assertion follows by Lemma 1.

(ii) Under the additional hypotheses in (ii) the result in (i) can be written as

$$\lim_{n\to\infty} \mathrm{P}\Big\{\frac{W_n}{a(m^n)} \le x\Big\} = \psi(x^{-a}), \quad x \in \mathrm{R}, \tag{19}$$

with $a(n) = n^{1/a}(L_1(n^{1/a}))^{1/a}$ where the s.v.f. $L_1(\cdot)$ is the one defined above, see e.g. [9], Prop.1.11.

In addition, by (3) one can obtain

$$\mathrm{E}Z_n \sim Mm^n, \quad n \to \infty, \tag{20}$$

where $M = \mu(1 + \sum_{k=0}^{\infty} \Phi_k(0)/m^{k+1}) \in (0, \infty)$.

Now, similarly to (8) using (20) we obtain that for $j \ge 1$

$$\mathrm{P}\Big\{\frac{W_n}{a(m^n)} > j\Big\} \le (1 - F(ja(m^n)))\mathrm{E}Z_{n-1}$$

$$= \frac{1 - F(ja(m^n))}{1 - F(a(m^n))} Mm^n(1 - F(a(m^n))).$$

For fixed $\varepsilon > 0$ and large n it follows from [9], Prop. 0.8 (ii) that

$$\frac{1 - F(ja(m^n))}{1 - F(a(m^n))} \leq (1+\varepsilon) j^{-a+\varepsilon}.$$

On the other hand by Theorem 1.5.12 in [1]

$$\limsup_{n\to\infty} m^n (1 - F(a(m^n)) = 1.$$

Therefore,

$$\limsup_{n\to\infty} \sum_{j=N+1}^{\infty} \mathrm{P}\Big\{\frac{W_n}{a(m^n)} > j\Big\} \leq M \sum_{j=N+1}^{\infty} j^{-a+\varepsilon},$$

and hence

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} \mathrm{P}\Big\{\frac{W_n}{a(m^n)} > j\Big\} = 0, \tag{21}$$

by the convergence of $\sum_{j=N+1}^{\infty} j^{-a+\varepsilon}$ for $a > 1$ and $0 < \varepsilon < a - 1$.

Now, from (19), (21), and Lemma 2 with $V_n = W_n/a(m^n)$ we obtain

$$\lim_{n\to\infty} \mathrm{E}\Big(\frac{W_n}{a(m^n)}\Big) = \int_0^{\infty} (1 - \psi(x^{-a}))dx,$$

which is equivalent to the assertion in (ii).

6. Concluding remarks. The comparison of the obtained results with those for the maximum number of offspring in the BGW processes without immigration from [8] is interesting. In the supercritical case it is not surprising that the immigration has little effect on the limiting behavior of W_n. Theorem 2 differs from the corresponding results in [8] only in the form of the Laplace transform $\psi(u)$. In the subcritical and critical cases the mechanism of immigration at zero replaces the conditioning on non–extinction. Instead of the conditional limit theorems given in [8], here we obtain unconditional ones under immigration at zero which acts as a reflecting barrier. Theorem 2 is a new result (see also [6], Theorem 1) which differs significantly from the limit theorem for W_n in the critical process without immigration. Due to the nonlinear normalization, the study of the limiting behavior of EW_n in this case needs additional efforts. To what extend the immigration rate can change the behavior of W_n as the reproduction process is "close", in some sense, to a critical one, is another open problem.

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МАКСИМАЛЕН БРОЙ ПОТОМЦИ В НЕКРИТИЧЕСКИ РАЗКЛОНЯВАЩ СЕ ПРОЦЕС СЪС ЗАВИСЕЩА ОТ СЪСТОЯНИЕТО ИМИГРАЦИЯ

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Разглежда се броят W_n на потомците на най-продуктивната частица от $n-1$-то поколение в разклоняващ се процес с имиграция, зависеща от състоянието на процеса. Доказани са гранични теореми за W_n и EW_n. Резултатите са получени като са използувани методите от [8] и известните резултати за общата численост на популацията в разклоняващ се процес с имиграция в нулата.