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SOLVING SOME PROBLEMS IN THE THEORY OF P. I. ALGEBRAS BY *MATHEMATICA*

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In the paper some applications of the system for computer algebra *Mathematica* are shown in solving important problems in the theory of P.I. algebras. They concern an approach introduced by Formanek and Bergman for investigating matrix identities by means of commutative algebra. An algorithm is shown for constructing a special associative polynomial using its commutative correspondent. This algorithm is realized in a package in *Mathematica* and applications of the package are shown for some investigations made.

The investigation of polynomial identities in matrix algebras with involution is accompanied by many computational difficulties. In this paper we introduce a possible approach for overcoming some of these difficulties by the system for computer algebra *Mathematica*.

In [8] we considered in details the idea of recursive construction of different in type polynomials from the free associative algebra $K\langle X\rangle$ over a field K of characteristic 0 aiming to find which of them are identities in the matrix algebra $M_n(K)$.

Now we want to show a way of using the system for computer algebra *Mathematica* for the following purpose:

It is very effective to describe matrix identities by means of commutative algebra. It is an approach introduced by Formanek [3] and Bergman [1] and used by Ts. Rashkova and V. Drensky [2] in the case of weak polynomial identities for Lie algebras.

To a polynomial in commuting variables

(1) $$g(t_1,\dots,t_{n+1})=\sum\alpha_p t_1^{p_1}\dots t_{n+1}^{p_{n+1}}\in K[t_1,\dots,t_{n+1}]$$

we relate a polynomial $v(g)$ from the free associative algebra $K\langle x,y_1,\dots,y_n\rangle$

(2) $$\begin{aligned} v(g) &= v(g)(x,y_1,\dots,y_n)\\ &= \sum\alpha_p x^{p_1}y_1x^{p_2}y_2\dots x^{p_n}y_nx^{p_{n+1}}.\end{aligned}$$

Any multilinear in $y_1,\dots,y_n$ polynomial $f(x,y_1,\dots,y_n)$ can be written as

(3) $$\begin{aligned} f(x,y_1,\dots,y_n) = \sum\ &\beta_i\, v(g_i)(x,y_{i_1},\dots,y_{i_n}),\\ &\beta_i\in K,\ g_i\in K[t_1,\dots,t_{n+1}].\end{aligned}$$

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A theorem of Bergman [1] gives a necessary and sufficient condition polynomials of type (2) and (3) to be identities in a given matrix algebra describing the corresponding polynomials of type (1). It states:

Theorem 1. [1, Section 6, (27)] (i) *The polynomial* $v(g)(x, y_1, \ldots, y_n)$ *from* (2) *is an identity for* $M_n(K)$ *if and only if*

$$\prod_{1 \le p < q \le n+1} (t_p - t_q)$$

divides $g(t_1, ..., t_{n+1})$ *from* (1).

(ii) *The polynomial* $f(x, y_1, \ldots, y_n)$ *from* (3) *is an identity for* $M_n(K)$ *if and only if every summand* $v(g_i)$ *is also an identity for* $M_n(K)$.

A goal of series of investigations was finding an analogue of this theorem in the case when the matrix algebra is equipped with the symplectic involution $*$. Then the algebra is of even order and $*$ is defined in $M_{2n}(K, *)$ by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix},$$

where A, B, C, D are $n \times n$ matrices and t is the usual transpose.

Due to [7] a sensible way of searching such an analogue is considering only $*$-identities in skew symmetric to the involution variables. An analogue of the theorem was found in the considered case and it is stated in [6].

Let $R = (R, *)$ be an associative algebra with involution $*$. An element $f(x_1, \ldots, x_m)$ from the free associative algebra with involution $K\langle X \cup X^* \rangle$ is a $*$-polynomial identity for $(R, *)$ if $f(r_1, \ldots, r_m) = 0$ for all $r_1, \ldots, r_m \in R$. Since $(R, *) = R^+ \oplus R^-$, where $R^\pm = \{r \in R \mid r^* = \pm r\}$, we can replace $K\langle X \cup X^* \rangle$ with $K\langle Y \cup Z \rangle$, where Y and Z are, respectively, sets of symmetric and skew symmetric variables.

Now we turn to the polynomials of type (1) and (2).

The first approach to construct polynomials of type (2) was more or less the direct one, used when applying *Mathematica* in illustrating partial cases from the theorems in [4,5].

Let we have the polynomial g from type (1) of low degree and its corresponding polynomial of type (2) $v(g)(x, y_1, \ldots, y_n) \in K\langle x, y_1, \ldots, y_n \rangle$ constructed directly. It is easy to be seen that $t_i g$ corresponds to $v(g)(x, y_1, \ldots, y_{i-1}, xy_i, \ldots, y_n)$ for $i = 2, ..., n-1$, to $xv(g)$ for $i = 1$ and to $v(g)x$ for $i = n$. Thus all polynomials of type (1) having g as a factor correspond to polynomials of type (2) which are consequences of higher degrees of $v(g)$ and their construction becomes immediate.

This approach however works only for the transfer (1) $\to$ (2).

Another view on the point is connected with the special advantages of the considered system itself [9]. If we have the polynomial g in order to get the corresponding polynomial $v(g)$ we make the substitutions $t_i^{p_i} \to x^{p_i} y_i$ for $i = 1, ...k-1$ and $t_k^{p_k} \to x^{p_k}$. We use the operator **ReplaceAll** in his short form /. to realize this substitution.

This approach works in both cases (1) $\longleftrightarrow$ (2) but it is very far from time effectiveness as we have to go through all partitions $(p_{i_1}, \ldots, p_{i_n})$ in (1) and in (2).

The third approach is more formalized and it is applied to any polynomial of type (1) not needing a direct construction of any part of it at the beginning.

Taking into account the peculiarities and the possibilities of the system *Mathematica* we apply an algorithm, the steps of which are the following ones:

1. For the variables $t_1, t_2, ..., t_n$ of the polynomial (1) we construct the form

$$t_1 * y_1 * t_2 * y_2 * ... * t_{n-1} * y_{n-1} * t_n, \tag{4}$$

where $*$ is noncommutative multiplication.

2. For every term of the polynomial g we extract the degree of the corresponding t_i for every i and put it as degree of t_i in the considered form.

3. The result of 2. is multiplied by the coefficient of the corresponding term.

This algorithm is realized in the package "Pitita". It consists of four modules two of which construct the standard polynomial

$$s_n(x_1, ..., x_n) = \sum_{\sigma \in Sym(n)} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} ... x_{\sigma(n)}$$

and Kapelli's polynomial

$$d_k(x_1, ..., x_k; y_0, ..., y_k) = \sum_{\sigma \in Sym(k)} (-1)^\sigma y_0 x_{\sigma(1)} y_1 ... x_{\sigma(k)} y_k.$$

The other two modules deal with the realization of the stated algorithm.

The polynomial (2) is constructed by the function **Tita**, which has two formats: **Tita**[⟨*polynomial*⟩] and **Tita** [⟨*polynomial*⟩ , ⟨*operation*⟩] .

In the first format the operator is supposed to be

NonCommutativeMultiply

denoted by $**$, while in the next format the parameter ⟨*operation*⟩ could be any built-in or user-defined operation.

In the function **Tita** the form from (4) is formed for a single monome and if g has several terms the function **TitaTerm** is applied to any of them.

TitaTerm is the function realizing steps 2. and 3. of the algorithm and it is called in the following form:

$$\mathbf{TitaTerm}[\langle g - term\rangle, \langle form\ (4)\ \ for\ substitution\rangle, \\ \langle list\ of\ the\ variables\ of\ g\rangle]$$

In the form (4) this function substitutes the variables of the polynomial g with their degrees of the parameter $\langle g - term\rangle$ by the operator

$$\mathbf{MapThread}[\#1 \to \#2\&, \{vars, \exp s\}]\,.$$

After that on the places of the names of the original variables the sign x is put by the operator

$$\mathbf{Map}[\#1 \to ToExpression[\text{“}x\text{”}], vars].$$

At the end the obtained monomes are multiplied by the corresponding coefficients of the $\langle g - term\rangle$.

The package is tested by many examples the most serious of which is the polynomial of degree 15 appearing in the case $n = 3$ from [7, Theorem 2].

The use of the package "Pitita" could be demonstrated in the system *Mathematica*.

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РЕШАВАНЕ НА НЯКОИ ВЪПРОСИ ОТ ТЕОРИЯТА НА АЛГЕБРИТЕ С ПОЛИНОМНИ ТЪЖДЕСТВА ЧРЕЗ *MATHEMATICA*

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В статията са посочени някои приложения на системата за компютърна алгебра *Mathematica* за решаване важни въпроси от теорията на алгебрите с полиномни тъждества. Те се отнасят до подход, въведен от Форманек и Бергман, за изследване матрични тъждества чрез средствата на комутативната алгебра. Посочен е алгоритъм за конструиране на специален асоциативен полином по съответния му полином на комутиращи променливи. Този алгоритъм е реализиран в пакет на *Mathematica* и са посочени приложения на създадения пакет при направени изследвания.