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CHARACTERISATION OF A FOUR-DIMENSIONAL RIEMANNIAN MANIFOLDS BY CHARACTERISTIC COEFFICIENTS OF JACOBI OPERATOR

Veselin Totev Videv

In the present paper we investigate 4-dimensional Riemannian manifolds (M, g) for which two of the characteristic coefficients of the Jacobi operator R_X are a pointwise constants at any point $p \in M$.

Let (M, g) be a four-dimensional Riemannian manifold with a metric tensor g and ∇ let be an uniquely Levi-Civita connection induced by a metric g. Let R be a curvature tensor of the manifold defined as bilinear mapping by the equality $R(x, y) = [\nabla_X, \nabla_Y] - \nabla_{[x,y]}$, where x, y are arbitrary tangent vectors in the tangent space M_p at a point $p \in M$, and $[.,.]$ are Lie brackets. The *Jacobi operator* R_X is a symmetric linear operator of the tangent space M_p at a point $p \in M$ defined by the equality $R_X(u) = R(u, X, X)$ [5]. For the matrix of R_X with respect to an arbitrary orthonormal Lorentzian basis in M_p of type (1) we have the entries $a_{ij} = R(e_i, X, X, e_j)$; $i, j = 1, 2, \ldots, n$. Since X is an eigenvector of R_X with a corresponding eigenvalue 0, then the characteristic equation $\det(a_{ij} - cg_{ij}) = 0$ of R_X can be represented by $\sum_{k=0}^{n} (-1)^k J_k c^{n-k}$, where $J_0 = 1$, $J_n = 0$; $J_i = J_i(p; X)$, $i = 1, 2, \ldots, n$. Because of $J_1(p; X) = \operatorname{trace} R_X = \rho(X)$, where ρ is the Ricci tensor on M, then $\operatorname{trace} R_X$ is a pointwise constant on the manifold (by $\dim M \geq 3$) if and only if (M, g) is an Einstein Lorentzian manifold. The problem about a global constancy of the eigenvalues of RX was created in Riemanian geometry from Bob Osserman [5] and the manifolds satisfying this hypothesis was called *globally Oserman manifolds* [6] and it was proved from Chi [1] that (M, g) *is a globally Osserman manifolds if and only if* (M, g) *locally is a rank one symmetric space or* (M, g) *is flat* when $\dim M = 4$ or m is odd or if $m \equiv 2 (\mathrm{mod}\, 4)$. Manifolds for which eigenvalues of R_X are pointwise constants on M are called *point-wise Oserman manifolds* and was investigated in [6]. In the case $\dim M = 4$ the Jacobi operator R_X has the following characteristic equation: $c(c^3 - J_1 c^2 + J_2 c - J_3) = 0$. Let p be a fixed point of M and let e_1, e_2, e_3, e_4 be an orthonormal basis in the tangent space M_p. Using the characteristic equation of the Jacobi operator $R_{ae_1+be_2}$ for any real numbers a, b $(a^2+b^2=1)$ we obtain

$$J_2(p; ae_1+be_2) = a^4(K_{13}K_{14} - R_{3114}^2) + b^4(K_{23}K_{24} - R_{3224}^2)$$
$$+2a^3b(K_{13}R_{1442} + K_{14})R_{1332} - (R_{3124} + R_{3214})_{3114})$$
$$+a^2b^2(K_{13}K_{24} + 4R_{1332}R_{1442} + K_{14}K_{23} - (R_{3124} + R_{3214})^2 - 2R_{3114}R_{3224})$$
$$(1) \qquad +2ab^3(K_{24}R_{1332} + K_{23}R_{1442} - (R_{3124} + R_{3214})R_{3224})$$
$$+2ab(K_{12}(R_{1332} + R_{1442}) + R_{2113}R_{1223} + R_{2114}R_{1224})$$
$$+a^2(K_{12}K_{13} + K_{12}K_{14} - R_{2113}^2 - R_{2114}^2)$$
$$+b^2(K_{12}K_{23} + K_{12}K_{24} - R_{1223}^2 - R_{1224}^2),$$

$$J_3(p; ae_1+be_2) =$$
$$a^4(2R_{2113}R_{3114}R_{2114} + K_{12}K_{13}K_{14} - K_{12}R_{3114}^2 - K_{13}R_{2114}^2 - K_4R_{2113}^2)$$
$$+2a3^b(R_{2114}R_{2113}(R_{3124} + R_{3214}) - R_{2113}R_{3114}R_{1224} - R_{1223}R_{3114}R_{2114}$$
$$+K_{12}K_{14}R_{1332} + K_{12}K_{13}R_{1442} - K_{12}(R_{3124} + R_{3214})R_{3114}$$
$$-R_{1332}R_{2114}^2 + K_{13}R_{2114}R_{1224} + K_{14}R_{2113}R_{1223} - R_{1442}R_{2113}^2)$$
$$+a^2b^2(K_{12}K_{13}K_{24} + 4K_{12}R_{1442} - 2R_{1224}R_{2113}(R_{3124} + R_{3214})$$
$$+K_{12}K_{14}K_{24} + 2R_{1223}R_{3224}R_{2114} - 2R_{1223}R_{3114}R_{1224}$$
$$-2(R_{3124} + R_{3214})R_{1223}R_{2114} - K_{23}R_{2114}^2 + 4R_{1332}R_{2114}R_{1224} - K_{13}R_{1224}^2$$
$$-K_{12}(R_{3124} + R_{3214})^2 - 2K_{12}R_{3114}R_{3224} - K_{24}R_{2113}^2 + 4R_{1442}R_{2113}R_{1223}$$
$$-K_{14}R_{1223}^2) + 2ab^3(K_{12}K_{23}R_{1442} + K_{12}K_{24}R_{1332} - R_{2113}R_{3224}R_{1224}$$
$$-R_{1224}R_{1223}(R_{3124} + R_{3214}) - R_{2114}R_{1223}(R_{3124} + R_{3214})$$
$$+K_{23}R_{2114}R_{1224} - R_{1224}^2R_{1332} - K_{12}R_{3224}(R_{3124} + R_{3214})$$
$$+K_{24}R_{1223}R_{2113} - R_{1223}^2R_{1442})$$
$$+b^4(K_{12}K_{23}K_{24} - 2R_{1223}R_{3224}R_{1224} - K_{12}R_{3224}^2 - K_{23}R_{1224}^2 - K_{24}R_{1223}^2).$$

Suppose e_1, e_2, e_3, e_4 are eigenvectors of the Jacobi operator R_{e_1}. Then we have equalities $R_{2113} = R_{2114} = R_{3114} = 0$. From the condition $J_2(p; ae_1+be_2) = J_2(p; -ae_1+be_2)$ and (1) we obtain the system

$$(K_{12} + K_{13})R_{1442} + (K_{12} + K_{14})R_{1332} = 0,$$
$$(2) \qquad (K_{12} + K_{13})R_{1443} + (K_{13} + K_{14})R_{1223} = 0,$$
$$(K_{12} + K_{14})R_{1334} + (K_{13} + K_{14})R_{1224} = 0.$$

Also from the condition $J_3(p; ae_1+be_3) = J_3(p; -ae_1+be_3)$ and according to (1) we have

$$K_{12}(K_{13}R_{1442} + K_{14}R_{1332}) = 0, \qquad K_{13}(K_{14}R_{1223} + K_{12}R_{1443}) = 0,$$
$$K_{14}(K_{12}R_{1334} + K_{13}R_{1224}) = 0.$$

Suppose $J_3(p; e_1) = K_{12}K_{13}K_{14}$ is different from zero. Then all eigenvalues K_{12}, K_{13},

K_{14} of R_{e_1} are different from zero. From the last system we receive:

$$
\begin{aligned}
&K_{12}R_{1334} + K_{13}R_{1224} = 0,\\
&K_{12}R_{1443} + K_{14}R_{1223} = 0,\\
&K_{13}R_{1442} + K_{14}R_{1332} = 0.
\end{aligned}
\tag{3}
$$

Now from (2) and (3) we obtain following two systems:

$$
(K_{12} - K_{13})R_{1334} = 0, \quad (K_{13} - K_{14})R_{1442} = 0, \quad (K_{12} - K_{14})R_{1223} = 0; \tag{4}
$$

$$
R_{1442} + R_{1332} = 0, \quad R_{1443} + R_{1223=0}, \quad R_{1334} + R_{1224} = 0. \tag{5}
$$

From the eigenvalues of the Jacobi operator R_{e_1} we have the following possibilities:

$$
K_{12} = K_{13} = K_{14}; \tag{6}
$$

$$
K_{12} \neq K_{13} \neq K_{14} \tag{7}
$$

(8) Two of eigenvalues of R_{e_1} are equal for example $K_{12} = K_{13}$ and also $K_{12} = K_{13} \neq K_{14}$.

Suppose (6) holds. Then from arbitrarity of e_1 it follows that (M, g) is a space of constant sectional curvature K_{12} at a fixed point $p \in M$, i.e. $R(x, y, z) = \mu(g(y, z)x - g()y)$, for any $x, y, z \in M_p$.

Suppose (7) is hold. Then from the system(4) we obtain

$$
R_{1jjk} = 0, \qquad j \neq k \quad (j, k = 1, 2, 3, 4), \tag{9}
$$

and then from the equalities (3), (4) and (10) it follows that

$$
\begin{aligned}
&J_2(p; e_1) = K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14},\\
&J_3(p; e_1) = K_{12}K_{13}K_{14};\\
&J_2(p; e_2) = K_{12}K_{23} + K_{12}K_{24} + K_{23}K_{24} - R^2_{3224},\\
&J_3(p; e_2) = K_{12}K_{23}K_{24} - K_{12}R^2_{3224};\\
&J_2(p; e_3) = K_{13}K_{23} + K_{13}K_{34} + K_{23}K_{34} - R^2_{2334},\\
&J_3(p; e_3) = K_{13}K_{23}K_{34} - K_{13}R^2_{2334};\\
&J_2(p; e_4) = K_{14}K_{24} + K_{14}K_{34} + K_{24}K_{34} - R^2_{2443},\\
&J_3(p; e_4) = K_{12}K_{24}K_{34} - K^R_{14}2_{2443}.
\end{aligned}
\tag{10}
$$

From the condition $J_3(p; e_1) = J_3(p; e_2)$ we have $K_{12}K_{13}K_{14} = K_{12}(K_{23}K_{24} - R^2_{2334})$ and since $K_{12} \neq 0$, then from the last equality we have:

$$
K_{13}K_{14} = K_{23}K_{24} - R^2_{3224}. \tag{11}
$$

From the condition $J_2(p; e_1) = J_2(p; e_2)$ we receive $K_{12}K_{13} + K_{12}K_{14} + K_{13}K_{14} = K_{12}K_{23} + K_{12}K_{24} + K_{23}K_{24} - R^2_{3224}$. Now from the last equality and (11) it follows that $K_{12}(K_{13} + K_{14}) = K_{12}(K_{23} + K_{24})$ and since $K_{12} \neq 0$ then we obtain $K_{13} + K_{14} = K_{23} + K_{24}$. Further using the conditions $J_2(p; e_1) = J_2(p; e_3) = J_2(p; e_4)$, $J_3(p; e_1) = J_3(p; e_3) = J_3(p; e_4)$ and (10) we have $K_{12} + K_{14} = K_{23} + K_{34}$, $K_{12} + K_{13} = K_{24} + K_{34}$. Thus the pointwise conditions of the characteristic coefficients $J_2(p; e_1)$ and

$J_3(p;e_1)$ give us the system

$$\begin{aligned} K_{12}+K_{13}&=K_{24}+K_{34},\\ K_{12}+K_{14}&=K_{23}+K_{34},\\ K_{13}+K_{14}&=K_{23}+K_{24}, \end{aligned} \tag{12}$$

and from this system we obtain directly

$$K_{12}=K_{34}, \quad K_{13}=K_{24}, \quad K_{14}=K_{23}. \tag{13}$$

From here and from the condition

$$J_2(p;e_1)=J_2(p;e_2)=J_2(p;e_3)=J_2(p;e_4),$$

it follows that

$$R_{ijjk}=0, \quad i\neq j\neq k, \quad (i,j,k=1,2,3,4). \tag{14}$$

The equalities (13) and (14) means that eigenvalues of the Jacobi operator R_{e_1} formed a Singer-Thorpe basis in the tangent space M_p. Following [3] we use the standard denoting: $K_{12}=K_{34}=1$, $K_{13}=K_{24}=2$, $K_{14}=K_{23}=3$, $R_{1234}=1$, $R_{1342}=2$, $R_{1423}=3$. Then from the characteristic equation of $R_{ae_1+be_2}$ $(a^2+b^2=1;\ a,b\in R)$ we have:

$$(c-\lambda_1)(c^2-(\lambda_2+\lambda_3)c+\lambda_2\lambda_3+a^2b^2((\lambda_2-\lambda_3)^2+(\mu_2-\mu_3)^2))=0,$$

and from here it follows that

$$J_2(p;ae_1+be_2)=\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3+a^2b^2((\lambda_2-\lambda_3)^2+(\mu_2-\mu_3)^2).$$

If the characteristic polynomial of the Jacobi operator R_{e_1} is a point-wise constant, then we obtain the equality $\lambda_2-\lambda_3=+(\mu_2-\mu_3)$. Analogously from the pointwise conditions of the characteristic coefficients $J_2(p;ae_1+be_3)$ and $J_2(p;ae_1+be_4)$ we obtain two equalities $\lambda_3-\lambda_1=+(\mu_3-\mu_1)$ and $\lambda_1-\lambda_2=+(\mu_1-\mu_2)$. Hence we have the system

$$\lambda_2-\lambda_3=\pm(\mu_2-\mu_3), \qquad \lambda_3-\lambda_1=\pm(\mu_3-\mu_1), \qquad \lambda_1-\lambda_2=\pm(\mu_1-\mu_2)$$

and according to the results in [3] we obtain that (M,g) is a pointwise constant at the fixed point p of M.

If we have (8), then using (7) we receive:

$$R_{1442}=R_{1443}=R_{1223}=R_{1332}=0 \tag{15}$$

and the characteristic equation of the Jacobi operator $R_{ae_1+be_2}$ with respect to the orthonormal basis ae_1+be_2, $-be_1+ae_2$, e_3, e_4, has the form

$$\begin{vmatrix} K_{12}-c & 0 & bR_{1224} \\ 0 & a^2K_{13}+b^2K_{23}-c & b^2R_{3224}+ab(R_{3124}+R_{3214}) \\ bR_{1224} & b^2R_{3224}+ab(R_{3124}+R_{3214}) & a^2K_{14}+b^2K_{24}-c \end{vmatrix}=0.$$

From here it follows that

$$\begin{aligned} J_3(p;ae_1+be_3)=K_{12}((a^2K_{13}+b^2K_{23})(a^2K_{14}+b^2K_{24})-(b^2R_{3224}+ab(R_{3124}+R_{3214})^2))\\ -b^2R_{1224}^2(a^2K_{13}+b^2K_{23}). \end{aligned}$$

According to our assumption this coefficient to be a pointwise constant we obtain $K_{23}R_{1334}$ $=K_{23}R_{1224}=0$. Then we have $R_{1334}=R_{1224}=0$ or $K_{23}=0$. In the first case we have the equalities (10)–(15) again and from these equalities it follows that (M, g) is a space of constant sectional curvature at a fixed point p.

In the case $K_{23} = 0$ from the pointwise condition $J_2(p; e_1) = J_2(p; e_2)$ we obtain that $K_{12}(K_{13} + K_{14}) - K_{13}K_{14} = K_{12}K_{24}$ and hence $K_{13}K_{14} = 0$. Then $J_3(p; e_1) = 0$ which contradict with our assumption $J_3(p; e_1) = K_{12}K_{13}K_{14} \neq 0$.

Suppose $J_3(p; e_1) = K_{12}K_{13}K_{14} = 0$. Now we have the following logistic possibilities:

a) $K_{12} = K_{13} = K_{14} = 0$ – then (M, g) is flat at a fixed point p.

b) $K_{12} \neq 0$, $K_{13} = K_{14} = 0$. Then $K_{12} = \operatorname{trace} R_{e_1} = \dfrac{\tau}{4}$, where is a scalar curvature on the manifold and hence the eigenvalues of the Jacobi R_{e_1} are constants.

c) Let $K12 \neq 0$, $K_{13} \neq 0$, $K_{14} \neq 0$. Then from (10) it follows that

$$K_{12}(K_{23}K_{24} - R^2_{3224}) = K_{12}(K_{23}K_{24} - R^2_{2334}) = K_{12}K_{24}K_{34} = 0,$$

and from here we have

$$K_{23}K_{24} - R^2_{3224} = K_{23}K_{24} - R^2_{2334} = K_{24}K_{34} = 0 \tag{16}$$

and (10) again. Hence

$$J_2 = K_{12}K_{13} = K_{12}(K_{23} + K_{24}) = K_{13}(K_{23} + K_{34}) = -R^2_{2443}. \tag{17}$$

From here we receive:

$$\begin{aligned} K_{12} &= K_{23} + K_{24}, \\ K_{13} &= K_{23} + K_{34}. \end{aligned} \tag{18}$$

If $K_{23}K_{24} = 0$, then at least one of the sectional curvature K_{23}, K_{24} is equal to zero. If $K_{23} = K_{24} = 0$, then all eigenvalues of the Jacobi operator R_{e_1} are equal at a fixed point p.

Suppose one of the sectional curvature K_{23}, K_{24} is different from zero, say K_{24}, then from (18) we have

$$K_{12} = K_{23}, \quad K_{13} = K_{23} + K_{24}, \tag{19}$$

and from here and (17) it follows that

$$R^2_{2334} = -K_{12}K_{23}.$$

From the last equality, (17) and (19) we obtain:

$$R^2_{2334} = K_{23}K_{24} = K_{13}(K_{12} - K_{23}) = K_{12}K_{13} - K^2_{23} = -R^2_{2443} - K^2_{12}.$$

Then $R^2_{2334} + R^2_{2443} + K^2_{12} = 0$ and from here it follows that $K_{12} = 0$ which contradict with a hypothesis $K_{12} \neq 0$. From the result above we can formulate

Theorem 1. *Let (M, g) be a 4-dimensional Riemannian manifold such that the characteristic coefficients $J_2(p; x)$ and $J_3(p; x)$ of the Jacobi operator R_X are a pointwise constants for any unit tangent vector $X \in S_pM$ and at any fixed point $p \in M$. If e_1, e_2, e_3, e_4 is orthonormal basis in the tangent space M_p we have one of the following possibilities:*

a) e_1, e_2, e_3, e_4 is a Singer Thorpe basis such that $\lambda_1 = \lambda_2 = \lambda_3 = 0$,

b) e_1, e_2, e_3, e_4 is a Singer Thorpe basis such that $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$,

c) e_1, e_2, e_3, e_4 is an arbitrary orthonormal basis such that $K_{12} = \dfrac{\tau}{4}$, $K_{13} = K_{14} = 0$, *d)* e_1, e_2, e_3, e_4 *is a Singer Thorpe basis such that*

$$\lambda_2 - \lambda_3 = +(\mu_2 - \mu_3), \lambda_3 - \lambda_1 = +(\mu_3 - \mu_1), \lambda_1 - \lambda_2 = +(\mu_1 - \mu_2)$$

and at least two of the invariants $\lambda_1, \lambda_2, \lambda_3$ *are different.*

Theorem 2. *Let* (M, g) *be a four-dimensional Riemannian manifold such that the characteristic coefficients* $J_2(p; x)$ *and* $J_3(p; x)$ *of the Jacobi operator* R_X *are a pointwise constants for each unit tangent vector* $X \in M_p$ *and at any fixed point* $p \in M$*. Then* (M, g) *locally is almost every where one of the following types of the manifolds:*

a) a flat manifold,

b) a space of constant sectional curvature,

c) a pointwise Osserman manifold.

Proof. Let the set of all points on M is such that the number of eigenavlues of the Jacobi operator R_{e_1} is a locally constant. Because of R_{e_1} is a symmetric linear operator, then this set is almost everywhere open and dense on M [2]. Because of $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the Jacobi operator R_{e_1}, then from Theorem 1 we have one of the possibilities a)–d). If we have a) and b), then (M, g) is respectively a flat manifold and a space of constant sectional curvature on Ω. If c) holds, then (M, g) locally is a globally Osserman manifold which contradict the result of Chi [1] that in this case the eigenvalues of Jacobi operator R_X are equal to 1 and $\frac{1}{4}$. Finally if d) holds, then (M, g) is a pointwise Osserman manifold on Ω.

In our paper [7] we have obtained the following results:

Theorem 3. *Let* (M, g) *be a 4-dimensional Riemannian manifold such that the characteristic coefficients* $J_1(p; X)$ *and* $J_2(p, X)$ *of the Jacobi operator* R_X *are a point-wise constants. Then all eigen values* R_X *are also point-wise constants and the same is also true for the characteristic coefficien* $J_2(p; X)$*.*

Theorem 4. *Let* (M, g) *be a 4-dimensional Riemannian manifold such that the characteristic coefficients* $J_1(p; X)$ *and* $J_3(p, X) \neq 0$ *of the Jacobi operator* R_X *are a point-wise constants. Then all eigen values* R_X *are also point-wise constants and the same is also true for the characteristic coefficien* $J_2(p; X)$*.*

Finally we can formulate the main result:

Theorem 5. *Let* (M, g) *be a 4-dimensional Riemannian manifold such that two of the characteristic coefficients of a non-degenerated Jacobi operator* R_X *are a point-wise constants for any unit tangent vector* $X \in M_p$ *and at any fixed point* $p \in M$*. Then* (M, g) *locally is one of the following types of manifolds:*

a) a flat manifold,

b) a space of constant sectional curvature,

c) a pointwise Osserman manifold.

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Veselin Videv
Mathematics and Physics Department
Thracian University
6000 Stara Zagora
Bulgaria

ХАРАКТЕРИЗИРАНЕ НА ЧЕТИРИМЕРНИ РИМАНОВИ МНОГООБРАЗИЯ ЧРЕЗ ХАРАКТЕРИСТИЧНИТЕ КОЕФИЦИЕНТИ НА ОПЕРАТОРА НА ЯКОБИ

Веселин Тотев Видев

В представената статия изследваме чеетримерните Риманови многообразия (M, g) със свойството два от характеристичните коефициенти на оператора на Якоби R_X да са точково постоянни в произволна точка от многообразието.