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METRICAL CHARACTERIZATION AT A BASEPOINT OF A FOUR-DIMENSIONAL EINSTEIN LORENTZIAN MANIFOLDS BY JACOBI OPERATOR

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In the present paper we investigate a four-dimensional Einstein Lorentzian manifolds (M,g) such that the characteristic coefficient $J_1(p;X)$ of the Jacobi operator R_X is a pointwise constant on the manifold and $J_3(p;X)=0$ on M.

An n-dimensional Riemannian manifold (M,g) is called a Lorentzian manifold if at any point $p\in M$ the tangent space M_p is a vector space provided with a scalar product g of signature $(-,+,...,+)$ or $(+,...,+,-)$. The set of all tangent vector X such that $g(X,X)=1$ $(g(X,X)=-1)$ we denote by ${}^+S_pM$ $({}^-S_pM)$. Let $n=4$ and let

$$e_1,e_2,e_3,e_4 \quad (e_4\in {}^-S_pM) \tag{1}$$

be an arbitrary Lorentzian basis in the tangent space M_p at a point $p\in M$. We denote by $\wedge^2(M_p)$ the (6-dimensional) space of 2-vectors of M_p. The space $\wedge^2(M_p)$ is equipped with its standard inner product whose value on decomposable elements is given by $\widehat{g}(v_1\wedge v_2, w_1\wedge w_2)=\det g(v_i,w_j)$, $i,j=1,2,...,n$, where g and R are respectively the metric tensor and the curvature tensor on M. The curvature tensor $\Re$ is defined in $\wedge^2(M_p)$ by the equality

$$\Re(x\wedge y, z\wedge v)=R(x,y,z,v) \tag{2}$$

where $x,y,z,v\in M_p$. If (1) is an orthonormal Lorentzian basis in the tangent space M_p then

$$e_1\wedge e_2, e_1\wedge e_3, e_1\wedge e_4, e_3\wedge e_4, e_4\wedge e_2, e_2\wedge e_3 \tag{3}$$

is an orthonormal basis in the 2-vector space $\wedge^2(M_p)$ and it is a vector space of signature $(+,+,+,-,-,-)$. This assertion has been proven in [1]:

Theorem 1 (A. Z. Petrov). *Let (M,g) be a 4-dimensional Einstein Lorentzian manifold ($\rho=\lambda g$). Then at any point $p\in M$ there exist a Lorentzian basis of type (1) in the tangent space M_p such that the matrix of the curvature operator in 2-vector space $\wedge^2(M_p)$ with respect to an orthonoramal basis of type (3) has the form $\begin{pmatrix} M & N \\ -N & M\end{pmatrix}$, where the matrix M and N are one of the following three types:*

Type I. $M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$, $N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}$,

Type II. $M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2+1 & 0 \\ 0 & 0 & \alpha_2-1 \end{pmatrix}$, $N = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 1 \\ 0 & 1 & \beta_2 \end{pmatrix}$,

Type III. $M = \begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

According to this result and (2) the next theorem can be obtained:

Theorem 2. *Let (M, g) be a 4-dimensional Einstein Lorentzian manifold. Then at any point $p \in M$ there exist an orthonormal Lorentzian basis of type* (1) *with respect to which the components of the curvature tensor R are defined by one of the following three types formulas:*

(4) $$R_{1212} = -R_{3434} = \alpha_1, \quad R_{1313} = -R_{2424} = \alpha_2, \quad R_{2323} = -R_{1414} = \alpha_3;$$

(5) $$R_{1212} = -R_{3434} = \alpha_1, \quad R_{1313} = -R_{2424} = \alpha_2 + 1,$$
$$R_{2323} = -R_{1414} = \alpha_2 - 1, \quad R_{3114} = -R_{3224} = 1;$$

(6) $$R_{1212} = -R_{3434} = R_{1313} = -R_{2424} = R_{2323} = -R_{1414} = \alpha,$$
$$R_{3114} = -R_{3224} = 1, \quad R_{2443} = -R_{2113} = 1.$$

Remark 1. The basis of this property further will be mentioned as Petrov basis.

The Jacobi operator R_X is a symmetric linear operator of the tangent space M_p at a point $p \in M$ defined by $R_X(u) = R(u, X, X)$ [7]. The matrix of R_X with respect to an arbitrary orthonormal Lorentzian basis in M_p of type (1) has the entries $a_{ij} = R(e_i, X, X, e_j)$, $(i, j = 1, 2, ...n)$. Since X is an eigenvector of R_X corresponding with an eigenvalue 0, then the characteristic equation of R_X can be represented in the form $\sum_{k=0}^{n} (-1)^k J_k c^{n-k} = 0$, where $J_0 = 1$, $J_n = 0$; $J_i = J_i(p; X)$, $(i = 1, 2, ..., n)$. Because $J_1(p; X) = \operatorname{trace} R_X = \rho(X)$, where ρ is the Ricci tensor on M, then $\operatorname{trace} R_X$ is a pointwise constant on the manifold (by $\dim M \geq 3$) if and only if (M, g) is an Einstein Lorentzian manifold [6]. The problem about a global constancy of the eigenvalues of R_X was created in the Riemanian geometry from Bob Osserman [7]. The manifolds which satisfy this hypothesis was called *globally Oserman manifolds* [4]. It was proven from Chi [5] that (M, g) *is a globally Osserman manifolds iff* (M, g) *locally is a rank one symmetric space or* (M, g) *is flat by* $\dim M = 4$, *if m is odd, or if* $m \equiv 2(\mathrm{mod}\, 4)$ [5]. A manifolds for which eigenvalues of R_X are pointwise constants on M are called *pointwise Oserman* and they were investigated in details in [4]. The problem about a pointwise conctancy of the eigenvalues of R_X was transfered from N. Blazic, N. Bokan and P. Gilkey [3] in the Lorentzian geometry as a pointwise conctancy of the characteristic polynomial of Jacobi operator R_X for any tangent vector $X \in^{\pm} S_pM$ at any point $p \in M$ because in Lorentzian geometry R_X is not always diagonalizable. It was proven that (M, g) *is a pointwise Osserman manifold* (by $n \geq 3$) *iff* (M, g) *is a manifold of constant sectional curvature* [3]. Generalizing this result we proved that (M, g) *is a pointwise Osserman manifold* ($X \in^{\pm} S_pM$), $\dim M \geq 3$ *if and only if the characteristic coefficients* $J_1(p; X)$, $J_2(p; X)$ *or* $J_1(p; X)$, $J_3(p; X) \neq 0$ *are a pointwise constants at any point* $p \in M$ *and for any tangent vector* $X \in^{\pm} S_pM$ [8]. The case when $J_1(p; X)$ is a pointwise constant and $J_3(p; X) = 0$ we investigate in this note.

Lemma 1. *A 4-dimensional Lorentzian manifold* (M, g) *is a manifold of pointwise constant characteristic coefficients* $J_1(p; X)$ *and* $J_3(p; X) = 0$ *of the Jacobi operator* R_X *for any tangent vector* $X \in^{\pm} S_pM$ *if and only if at any point* $p \in M$ *for the invariants of a Petrov basis in* M_p *we have:*

$$\text{for type (4)} \quad \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0, \tag{7}$$

$$\text{or} \quad \alpha_1 \neq 0, \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0; \tag{8}$$

$$\text{for type (5)} \quad \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0; \tag{9}$$

$$\text{for type (6)} \quad \alpha = 0. \tag{10}$$

Proof. Let p be a point of M and let e_1, e_2, e_3, e_4 ($e_4 \in^{-} S_pM$) be a Petrov basis in M_p. If a and b are a real numbers such that $a^2 - b^2 = \varepsilon$, $\varepsilon = \pm 1$, then

$$ae_1 + be_4, be_1 + ae_4, e_2, e_3$$

is an orthonormal Lorentzian basis in M_p. If we have (4), then using the characteristic equations of $R_{ae_1+be_4}$, $R_{ae_2+be_4}$, $R_{ae_3+be_4}$ with respect to an orthonormal basis of type (10) we obtain:

$$\begin{aligned} J_3\left(p, ae_1 + be_4\right) &= \epsilon\alpha_3 \left(\alpha_1\alpha_2 - a^2b^2\left(\left(\alpha_1 - \alpha_2\right)^2 + \left(\beta_1 - \beta_2\right)^2\right)\right) = 0. \\ J_3\left(p, ae_2 + be_4\right) &= \epsilon\alpha_1 \left(\alpha_2\alpha_3 - a^2b^2\left(\left(\alpha_2 - \alpha_3\right)^2 + \left(\beta_2 - \beta_3\right)^2\right)\right) = 0, \\ J_3\left(p, ae_3 + be_4\right) &= \epsilon\alpha_2 \left(\alpha_3\alpha_1 - a^2b^2\left(\left(\alpha_3 - \alpha_1\right)^2 + \left(\beta_3 - \beta_1\right)^2\right)\right) = 0 \end{aligned} \tag{11}$$

and $J_3(p; e_1) = \alpha_1\alpha_2\alpha_3 = 0$. It is evident that at least one of the invariants α_i is equal to zero. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then according to the results in [1] we have that (M, g) is flat. If at least one of the invariants α_i is different from zero, say α_3, then $\alpha_1\alpha_2 = 0$. Now from (11) we obtain $a^2b^2((\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2) = 0$. From here it follows that $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2$ and using the first Bianci identity we obtain $\beta_3 = -2\beta_1$. Thus we have

$$\alpha_1 = \lambda,\ \alpha_2 = \alpha_3 = 0,\ \beta_2 = \beta_3,\ \beta_1 = -2\beta_2. \tag{12}$$

Let $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ be eigenvectors of $\Re$ and let $k_1, k_2, k_3, \bar{k}_1, \bar{k}_2, \bar{k}_3$ be the corresponding eigenvalues, where $k_j = \alpha_j + i\beta_j$ and $i^2 = -1$. If (12) are satisfied the matrix of curvature operator $\Re$ with respect to the basis of type (3) has the form:

$$\begin{pmatrix} \lambda-2i\beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda+2i\beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\beta_3 \end{pmatrix}.$$

Suppose the orthonormal basis $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ is decomposable, then an orthonormal basis v_1, v_2, v_3, v_4 ($v_4 \in {}^{-}S_pM$) exists in M_p with respect to which there are relations $R(v_1, v_2, v_2, v_1) = R(v_3, v_4, v_4, v_3) = \lambda - 2i\beta_2$, $R(v_1, v_3, v_3, v_1) = R(v_2, v_3, v_3, v_2) = -R(v_2, v_4, v_4, v_2) = -R(v_1, v_4, v_4, v_1) = i\beta_2$. Now using the characteristic equations of Jacobi operators $R_{v_1}, R_{v_2}, R_{v_3}, R_{v_4}$ with respect to the basis v_1, v_2, v_3, v_4 we obtain that R_{v_1} has eigenvalues $\dfrac{\lambda-2i\beta_2}{g(v_2,v_2)}$, $\dfrac{2i\beta_2}{g(v_3,v_3)}$, $\dfrac{2i\beta_2}{g(v_4,v_4)}$, R_{v_2} has eigenvalues $\dfrac{-2i\beta_2}{g(v_1,v_1)}$, $\dfrac{2i\beta_2}{g(v_3,v_3)}$, $\dfrac{2i\beta_2}{g(v_4,v_4)}$, R_{v_3} has eigenvalues $\dfrac{2i\beta_2}{g(v_1,v_1)}$, $\dfrac{2i\beta_2}{g(v_3,v_3)}$, $\dfrac{-2i\beta_2}{g(v_4,v_4)}$ and R_{v_4} has eigenvalues $\dfrac{\lambda+2i\beta_2}{g(v_1,v_1)}$, $\dfrac{-2i\beta_2}{g(v_2,v_2)}$, $\dfrac{2i\beta_2}{g(v_3,v_3)}$. From the hypothesis $J_3(p; e_1) = J_3(p; e_2) = J_3(p; e_3) = J_3(p; e_4) = 0$ it follows that $(\lambda - 2i\beta_2)2i\beta_2 = (\lambda + 2i\beta_2)2i\beta_2 = 0$. If $2i\beta_2 \neq 0$, then $\lambda - 2i\beta_2 = \lambda + 2i\beta_2 = 0$ and hence $\lambda = \beta_2 = 0$. Hence $\alpha_1 = \lambda = 0$ and because we have also $\alpha_2 = \alpha_3 = 0$, then according to the results in [1] we have that (M, g) is flat at a point p and it contradict with the assumption $\beta_2 \neq 0$. If $\beta_2 = 0$, then for the curvature component of R with respect to Petrov basis of type (1) in M_p we have (8), eventually by $\alpha_1 = 0$ we have (7).

Suppose we have (5) and let e_1, e_2, e_3, e_4 ($e_4 \in {}^{-}S_pM$) be Petrov basis in M_p and a and b are a real numbers such that $a^2 - b^2 = \varepsilon$, $\varepsilon = \pm 1$. Then for the characteristic equations of the Jacobi operators $R_{ae_1+be_4}$ and $R_{ae_2+be_4}$ with respect to a Lorentzian basis of type (10) we have respectively

$$(c + \varepsilon K_{23})(c^2 - c(K_{12} + K_{13}) + K_{12}K_{13} - a^2b^2((K_{12} - K_{13})^2 + (\beta_1 - \beta_2)^2)$$
$$+a^2((a^2K_{12} - b^2K_{23}) - c)) = 0,$$
$$(d + \varepsilon K_{13})(d^2 - d(K_{12} + K_{23}) + K_{12}K_{23} - a^2b^2((K_{12} - K_{23})^2 + (\beta_3 - \beta_1)^2)$$
$$+a^2((a^2K_{12} - b^2K_{13}) - d)) = 0.$$

From the condition $J_3(p; ae_1 + be_4) = J_3(p; ae_2 + be_4) = 0$ we have

$$(\alpha_2 - 1)\alpha_2^2 + (\alpha_2 + 1)9\beta_2^2 = 0, \quad (\alpha_2 + 1)\alpha_2^2 + (\alpha_2 - 1)9\beta_2^2 = 0,$$

and from this system we obtain (9). If we have (6), then $J_3(p; e_1) = \alpha^3 = 0$, $J_3(p; e_4) = -\alpha(\alpha^2 - 1) = 0$ and from here we obtain (10).

Theorem 3. *A 4-dimensional Lorentzian manifold (M, g) is a manifold of pointwise constant characteristic coefficients $J_1(p; X)$ and $J_3(p; X) = 0$ of the Jacobi operator R_X for any tangent vector $X \in {}^{\pm}S_pM$ if and only if at any point $p \in M$ the metrics of M is* *one of the following three types:*

(13) $$ds^2 = 0,$$

(14) *a decomposable metrics which is decompose to the quadratic forms:*

$$ds^2 = dx_1^2 + \cos^2(\sqrt{\lambda}x_1)dx_2^2 + dx_3^2 - \cos^2(\sqrt{\lambda}x_3)dx_4^2; \lambda > 0;$$

$$ds^2 = dx_1^2 + \mathrm{ch}^2(\sqrt{-\lambda}x_1)dx_2^2 + dx_3^2 - \mathrm{ch}^2(\sqrt{-\lambda}x_3)dx_4^2; \lambda > 0, \lambda = \mathrm{const};$$

(15) $$ds^2 = dx_1^2 + \mathrm{sh}^2(x_1 - x_4)dx_2^2 + \sin^2(x_1 - x_4)dx_3^2 - dx_4^2;$$

Remark 2. The metrics in (14) and (15) are given in a special coordinate system [1].

Proof. *The if part.* If e_1, e_2, e_3, e_4 ($e_4 \in {}^-S_pM$) is a Petrov basis in the tangent space M_p, at a point $p \in M$, then for the curvature components with respect to this basis we have one of the formulas (4)-(6). From our assumption $J_1(p; X)$ to be a pointwise constant and $J_3(p; X) = 0$, for any tangent vector $X \in {}^\pm S_pM$, for the invariants of a Petrov basis we have one of the possibilities (7)-(10). If (7) holds, then (M, g) is flat at p and we have (13). If (8) are satisfied, then (M, g) is a decomposable space with a metric given in a special coordinate system by the equalities (14) – these results follows from the investigations in [1]. If (9) are satisfied, then we have the following system of differential equations $\nabla_{E_t} R(E_i, E_j, E_k, E_s) = 0$, where E_i are a smooth vector fields defined in a neighbourhood U_p around a point $p \in M$ such that $E_{i|p} = e_i$, $i = 1, 2, 3, 4$. Finding conditions for the integralibility of these equations we differentiate once more and alternate by index of differentiation. Then using the Ricci equality [5] we obtain

(16) $$R_{slm[a}R^s_{b]gd} + R_{slm[g}R^s_{d]ab} = 0.$$

where [.] denote an alternation. Now from (5), (16) and putting $\lambda = 1$, $\mu = 4$ we receive $\alpha_1 = \beta_1 = 0$. Hence from (5) we have $2\alpha_1 = \tau$ again, where τ is a scalar curvature on M. Fixing indices in (16) and using the substitution $\lambda \leftrightarrow 2$, $\mu \leftrightarrow 4$, $\alpha \leftrightarrow 1$, $\beta \leftrightarrow 4$, $\gamma \leftrightarrow 1$, $\delta \leftrightarrow 2$ we obtain $\alpha_2 = 0$ and $\tau = 0$. In [1] was proven that an uniquely Einstein Lorentzian manifold (M, g) exists, such that $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$ and it is a Petrov space of maximal mobility of metrics given in a special coordinate system by (16). Finally we remark that if we have (10), then putting in (16) these expressions we obtain a contradiction and hence this case is impossible.

The only if part. If we have (13) for g, then (M, g) is flat at p and evidently (M, g) is an Einstien Lorentzian manifold also $J_3(p; X) = 0$ for any unit spacelike or timelike tangent vector $X \in M_p$, at any point $p \in M$. If for g we have (14), then (8) are satisfied either. Let y is an arbitrary unit spacelike or timelike tangent vector in M_p and let

(17) $$y = \sum_{i=1}^{6} a_i e_i, \quad a_1^2 + \ a_2^2 + \ a_3^2 - \ a_4^2 = \varepsilon, \quad \varepsilon = \pm 1,$$

where a_i are an arbitrary real numbers and e_1, e_2, e_3, e_4 ($e_4 \in {}^-S_pM$) be a Petrov basis in M_p. Then the characteristic equation of the Jacobi operator R_y is:

$$\mu^2(\mu^2 - \alpha_3\mu + (\alpha_1^2 - \alpha_4^2)(\alpha_2^2 + \alpha_3^2)(9\beta_1^2 + \alpha_3^2)) = 0.$$

Hence $J_3(p;y) = 0$ and $J_1(p;y) = 3$ is a pointwise constant which means (M, g) is an Einsteinium Lorentzian manifold. If for g we have (15), then we have also (9) and from (5) we obtain $K_{12} = 0$, $K_{13} = -1$, $K_{23} = 1$, $R_{3114} = -R_{3224} = 1$. If y is an arbitrary unit spacelike or timelike tangent vector in M_p given by (17), then

$$J_3(p;y) = \begin{vmatrix} (c+t)^2 & 0 & -a(c+t) & -a(c+t) \\ 0 & -(c+t)^2 & -b(c+t) & b(c+t) \\ -a(c+t) & -b(c+t) & a^2-b^2 & b^2-a^2 \\ -a(c+t) & -b(c+t) & a^2-b^2 & b^2-a^2 \end{vmatrix} = 0.$$

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ХАРАКТРЕИЗИРАНЕ НА МЕТРИКАТА В БАЗОВА ТОЧКА НА ЧЕТИРИМЕРНИ АЙНЩАЙНОВИ ЛОРЕНЦОВИ МНОГООБРАЗИЯ ЧРЕЗ ОПЕРАТОРА НА ЯКОБИ

Веселин Тодоров Видев

В представената статия изследваме четиримерни Айнщайнови Лоренцови многообразия със свойството характеристичният коефициент $J_1(p;X)$ на оператора на Якоби R_X да е точково постоянен, а характеристичният коефициент $J_3(p;X)$ на R_X да е равен на нула в произволна точка $p \in M$ и за произволен единичен неизотропен вектор $X \in M_p$.