МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 1999
MATHEMATICS AND EDUCATION IN MATHEMATICS, 1999
Proceedings of Twenty Eighth Spring Conference of
the Union of Bulgarian Mathematicians
Montana, April 5–8, 1999

ON A STABILIZING CONTROL DESIGN FOR A METHANE FERMENTATION PROCESS *

Mikhail Krastanov, Neli Dimitrova

A model of continuous methane fermentation process, described by a two-dimensional control system, is studied. We compute the static optimal point according to a given criterion and design a feedback control stabilizing the process around this point. Numerical results are also reported.

1. Introduction. We consider the following mathematical model of the continuous methane fermentation process [2, 6]:

$$\frac{dx}{dt} = \frac{k_1 s}{k_2 + s} x - ux \tag{1}$$

$$\frac{ds}{dt} = -k_3 \frac{k_1 s}{k_2 + s} x + u(s_{in} - s); \tag{2}$$

$$Q = k_4 \frac{k_1 s}{k_2 + s} x, \tag{3}$$

where $x = x(t)$ and $s = s(t)$ are state variables,

x is biomass concentration,
s is substrate concentration (i. e. output pollution level),
u is dilution rate (i. e. flow rate),
s_{in} is influent substrate concentration (i. e. input pollution level),
Q is methane gas flow rate,
$k_1,\ k_2,\ k_3$ are kinetic coefficients,
k_4 is a proportional coefficient.

The control input is the dilution rate u and the output is methane gas flow rate $Q = Q(u)$.

The above biological interpretation of $x,\ s,\ u,\ s_{in}$ and $k_i,\ i = 1, 2, 3, 4$, implies the following bounds for them:

$$x > 0,\ \ 0 < s < s_{in},\ \ 0 < u < \frac{k_1 s_{in}}{k_2 + s_{in}},\ \ Q > 0,\ \ k_i > 0,\ i = 1, \ldots, 4. \tag{4}$$

The aim of this note is to show how to synthesize a bounded control function u (with admissible values) which stabilizes the control system (1)–(2) in a suitable neighbourhood

*This work has been partially supported by the Bulgarian National Science Fund under grants No. MM 521/95, MM 612/96, MM 807/98.

of the static optimal point of this process. The static optimal point is defined and computed in Section 2. Section 3 presents a procedure for constructing a stabilizing feedback control. Numerical results are reported in Section 4.

2. The Static Optimal Point. The static model of the methane fermentation process is delivered from (1)–(2) by setting [2, 6]

$$\frac{dx}{dt} = 0, \quad \frac{ds}{dt} = 0.$$

Thus the static model is presented by the following system of nonlinear equations

$$\frac{k_1 s}{k_2 + s} - u = 0 \tag{5}$$

$$-k_3 \frac{k_1 s}{k_2 + s} x + u(s_{in} - s) = 0. \tag{6}$$

According to (4) denote

$$U = \left(0, \frac{k_1 s_{in}}{k_2 + s_{in}}\right).$$

For each $u \in U$ the nonlinear system (5)–(6) possesses an unique solution $(x^*(u), s^*(u))$, which can be found explicitly:

$$x^*(u) = \frac{s_{in} k_1 - (k_2 + s_{in})u}{k_3(k_1 - u)}, \quad s^*(u) = \frac{k_2 u}{k_1 - u}.$$

From $0 < \dfrac{k_1 s_{in}}{k_2 + s_{in}} < k_1$ it follows that $x^*(u) > 0$ and $0 < s^*(u) < s_{in}$ are satisfied. The set of all points

$$\{(x^*(u), s^*(u)) : u \in U\}$$

is called steady states of the dynamics (1)–(2). It is straightforward to check that if $(x^*(u), s^*(u))$ is a steady state point then the following relation holds true:

$$s^*(u) + k_3 x^*(u) = s_{in}.$$

By substituting $x = x^*(u)$ and $s = s^*(u)$ in the expression for Q in (3) we obtain the following representation of $Q(u)$:

$$Q(u) = \frac{k_4(s_{in} k_1 - (k_2 + s_{in})u)u}{k_3(k_1 - u)}.$$

Obviously $Q(u) > 0$ is fulfilled for $u \in U$. The function $Q(u)$ is called static characteristic of the dynamic process (1)–(2).

We have further

$$\frac{dQ(u)}{du} = \frac{k_4}{k_3} \frac{k_1^2 s_{in} - 2k_1(k_2 + s_{in})u + (k_2 + s_{in})u^2}{(k_1 - u)^2}.$$

Finding the solutions of $\dfrac{dQ(u)}{du} = 0$ reduces to solving the quadratic equation $k_1^2 s_{in} - 2k_1(k_2 + s_{in})u + (k_2 + s_{in})u^2 = 0$, which possesses an unique real root $u_0 \in U$,

$$u_0 = \frac{k_1(k_2 + s_{in}) - k_1\sqrt{k_2(k_2 + s_{in})}}{k_2 + s_{in}}. \tag{7}$$

Thus $u_0 \in U$ is the unique point where $Q(u)$ takes its maximum, that is $Q_{\max} = Q(u_0)$. The point $(x_0, s_0) = (x^*(u_0), s^*(u_0))$ is called static optimal point of (1)–(3).

3. Feedback Control Design. Let Ω be a compact neighbourhood of the static optimal point (x_0, s_0). Following [1] and [5] we shall introduce some notions. A bounded function $k : \Omega \to U$ will be called feedback. Any infinite sequence $\pi = \{t_i\}_{i=0}^{\infty}$ consisting of numbers

$$0 = t_0 < t_1 < t_2 < ...$$

with $\lim_{i\to\infty} t_i = \infty$ is called a partition of $[0, +\infty]$ and the number

$$d(\pi) := \sup_{i\geq 0}(t_{i+1} - t_i)$$

is its diameter. The trajectory associated to a feedback $k(x, s)$ and any given partition π is defined as the solution of (1)–(2) obtained by means of the following procedure (this procedure is borrowed from the theory of positional differential games and is systematically studied by Krasovskii and Subbotin in [4]): on every interval $[t_i, t_{i+1}]$ the initial state is measured, $u_i = k(x(t_i), s(t_i))$ is computed and then the constant control $u \equiv u_i$ is applied until time t_{i+1} is achieved, when a new measurement is taken.

Definition. *The feedback $k : \Omega \to U$ is said to stabilize asymptotically the system (1)–(2) at the point (x_0, s_0) if for every $\varepsilon > 0$ there exist $T > 0$, $\delta > 0$, a partition π with diameter not greater than δ such that for every point $(x, s) \in \Omega$ the corresponding trajectory of (1)–(2) is well defined on $[0, +\infty)$ and satisfies the following conditions:*
(a) $(x(t), s(t)) \in \Omega$ for every $t \geq 0$;
(b) $\|(x(t) - x_0,\ s(t) - s_0)\| < \varepsilon$ for every $t \geq T$ (here $\|(x, s)\|$ denotes the standard Euclidean norm in R^2).

After the coordinate change

$$\xi = \frac{x - x_0 - k_3(s - s_0)}{1 + k_3^2}$$
$$\eta = \frac{s - s_0 + k_3(x - x_0)}{1 + k_3^2}$$

the control system (1)–(2) can be written as follows:

$$\frac{d\xi}{dt} = f(\xi, \eta; u) \tag{8}$$
$$\frac{d\eta}{dt} = -u\eta, \tag{9}$$

where

$$f(\xi, \eta; u) = \frac{k_1(x_0 + \xi + k_3\eta)(s_0 - k_3\xi + \eta)}{k_2 + y_0 - k_3\xi + \eta} - u(x_0 + \xi). \tag{10}$$

Clearly, the point (x_0, s_0) is mapped into $(0, 0)$ in the new coordinate system.

Since the property asymptotic stability does not depend on the choice of the coordinate axes, we can study this property in some neighbourhood of the origin (using the new coordinates ξ and η).

Let be $B = \{(\xi, \eta) : |\xi| \leq r_1, |\eta| \leq r_2\}$ within $r_i > 0, i = 1, 2$. Let us assume that for any $(\xi,\eta)\in B$ the equation $f(\xi,\eta;u)=0$ has an unique solution $u^*(\xi,\eta)>0$. Define

$$u_{\min}=\min_{(\xi,\eta)\in B}u^*(\xi,\eta),\quad u_{\max}=\max_{(\xi,\eta)\in B}u^*(\xi,\eta).$$

Proposition 1. *Let us assume that:*
(i) for some $\delta>0$ the elements of the interval $I:=[u_{\min}-\delta,u_{\max}+\delta]$ are admissible values of the control, i. e. $I\subset U$;
(ii) $\max\{\frac{\partial f}{\partial u}(\xi,\eta;u):\ (\xi,\eta)\in B,\ u\in I\}<0.$
Then the control system (8)–(9) is asymptotically stabilizable at the origin $(0,0)$.

Remark. Proposition 1 holds true not only for systems for which f is determined from (10), but for every smooth f satisfying (i) and (ii).

Proof. Let us fix an arbitrary ε such that $0<\varepsilon<\min\{r_1,r_2\}$. Since I and B are compact, there exists a real $h>0$ such that for every integrable function $u:[0,h]\to I$ and every point $(\xi,\eta)\in B$ the solution of (8)–(9) starting from the point (ξ,η) is well defined. Let

$$m:=-\max\{\frac{\partial f}{\partial u}(\xi,\eta;u):\ (\xi,\eta)\in B,\ u\in I\}.$$

Assumption (ii) implies $m>0$. We set

$$M=\max\{|f(\xi,\eta;u)|:\ (\xi,\eta)\in B,\ u\in I\}.$$

Without loss of generality we may assume that $0<h\le\varepsilon/\max\{M,m\delta\}$. We define the partition π and the feedback $k=k(\xi,\eta)$ as follows: $\pi=\{ih\}_{i=0}^{\infty}$ and

$$k(\xi,\eta)\quad=\quad\begin{cases}u_{\max}+\delta, & \text{if } \xi>0,\\ u_{\min}-\delta, & \text{if } \xi<0,\\ u^*(\xi,\eta), & \text{if } \xi=0.\end{cases}$$

Let $(\xi_0,\eta_0)\in B$ be an arbitrary point and $(\xi(\cdot),\eta(\cdot))$ be the corresponding trajectory of the system (8)–(9). Taking into account the choice of h, this trajectory is well defined on $[0,h]$.

Consider first the case $\xi_0>0$. According to the definition of trajectory corresponding to the feedback $k=k(\xi,\eta)$, we have

$$\begin{aligned}f(\xi(\tau),\eta(\tau),k(\xi_0,\eta_0)) &= f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))-f(\xi(\tau),\eta(\tau);u^*(\xi(\tau),\eta(\tau)))\\ &= \frac{\partial f}{\partial u}(\xi(\tau),\eta(\tau);\zeta)(k(\xi_0,\eta_0)-u^*(\xi(\tau),\eta(\tau)))\\ &= \frac{\partial f}{\partial u}(\xi(\tau),\eta(\tau);\zeta)(u_{max}+\delta-u^*(\xi(\tau),\eta(\tau)))\le -m\delta\end{aligned}$$

for each $\tau\in[0,h]$. This presentation implies

$$\xi(h)\quad=\quad\xi_0+\int_t^{t+h}f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau\le\xi_0-m\delta h\ \le\ \xi_0\le r_1;$$

$$\xi(h)\quad=\quad\xi_0+\int_t^{t+h}f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau \tag{11}$$

$$\geq \int_t^{t+h} f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau \geq -Mh \geq -\varepsilon \geq -r_1.$$

Similarly, for $\xi_0 < 0$ we obtain

$$\xi(h) = \xi_0 + \int_t^{t+h} f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau \geq \xi_0 + m\delta h \geq \xi_0 \geq -r_1;$$

$$\xi(h) = \xi_0 + \int_t^{t+h} f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau \tag{12}$$

$$\leq \int_t^{t+h} f(\xi(\tau),\eta(\tau);k(\xi_0,\eta_0))d\tau \leq Mh \leq \varepsilon \leq r_1.$$

For $\xi_0 = 0$ we have

$$\xi(h) = \xi_0 + \int_t^{t+h} f(\xi(\tau),\eta(\tau);u^*(\xi_0,\eta_0))d\tau = \xi_0. \tag{13}$$

From (11), (12) and (13) it follows that $|\xi(h)| \leq r_1$. For $\eta(h)$ we have

$$|\eta(t)| \leq e^{-(u_{\min}-\delta)h}|\eta(0)| \leq e^{-(u_{\min}-\delta)h}r_2 < r_2.$$

Hence $(\xi(h),\eta(h)) \in B$. But then the trajectory of (8)–(9) will also be well defined on the interval $[h,2h]$ and will remain in B. Continuing in the same manner we shall obtain that the trajectory of (8)–(9) is defined on $[0,+\infty)$ and does not leave B. Moreover the inequalities (11)–(13) imply that $|\xi(t)| < \varepsilon$ for $t \geq T_1 := |\xi(0)|/(m\delta h)$ is valid. Since $k(\xi,\eta) \geq u_{\min} - \delta > 0$ for every $(\xi,\eta) \in B$ holds true, we obtain that

$$|\eta(t)| \leq e^{-(u_{\min}-\delta)t}|\eta(0)| < \varepsilon \text{ is also fulfilled for } t \geq T_2 := \frac{\ln r_2 - \ln \varepsilon}{u_{\min}-\delta} > 0.$$

Hence, for $t \geq \max\{T_1,T_2\}$ we shall have that $\|(\xi(t),\eta(t))\| \leq \varepsilon$ is satisfied. This completes the proof.

4. Numerical experiments. From the literature [3] and from practical experiments the following values for the the coefficients in the model (1)–(3) are known:

$$k_1 = 0.4;\ \ k_2 = 0.4;\ \ k_3 = 27.4;\ \ k_4 = 75;\ \ s_{in} = 3.$$

To demonstrate the theoretical results from the previous sections we use the computer algebra system *Maple V Release 3 for Windows* to perform the calculations and graphic visualizations.

Solving numerically the equation $\frac{dQ}{du}(u) = 0$ we obtain

$$u_0 = 0.262811318 \text{ and } Q_{\max} = Q(u_0) = 1.606882382.$$

The static optimal point (x_0,s_0) is given by the approximate values

$$x_0 = x^*(u_0) = 0.08152589862, \quad s_0 = s^*(u_0) = 0.7661903781;$$

obviously, $s_0 + k_3x_0 = s_{in}$ is fulfilled. According to Proposition 1 we have to determine a compact neighbourhood B of the origin $(0,0)$ such that $u^*(\xi,\eta) > 0$ holds for all $(\xi,\eta) \in B$. We define

$$B = \{(\xi,\eta) : -0.002 \leq \xi \leq 0.024,\ -0.0022 \leq \eta \leq 0.0009\}.$$

and find

$$u_{\min} = 0.03603272084, \qquad u_{\max} = 0.3524861030.$$

For $m = 0.07$ it follows $-\dfrac{\partial f}{\partial u} = x_0 + \xi \geq m > 0$ (see Proposition 1). Further we choose $\delta = 0.0004$, $\varepsilon = 0.0007$ and $h = 0.046$. We consider $t_i = ih$, $i = 0, 1, \ldots, n$, and initial conditions $\xi(0) = -0.001$, $\eta(0) = -0.000001$ from B. The substitution

$$x = x_0 + \xi + k_3\eta, \qquad s = s_0 - k_3\xi + \eta \tag{14}$$

delivers

$$x(0) = 0.08139849862, \quad s(0) = 0.768929378.$$

According to Proposition 1 we use an appropriate control $u = u_{\max} + \delta$ to compute $(\xi(t_1), \eta(t_1))$ and therefore $(x(t_1), s(t_1))$ according to (14); this process is being repeated changing the control u in the correct way. A worksheet in Maple was prepared to solve numerically the system (8)–(9) on each step t_i, $i = 1, 2, \ldots, n$. Thereby we used the procedure `dsolve` from the Maple library. Finally we computed $(x(t_i), s(t_i))$ according to (14) and $Q(t_i)$ by means of (3).

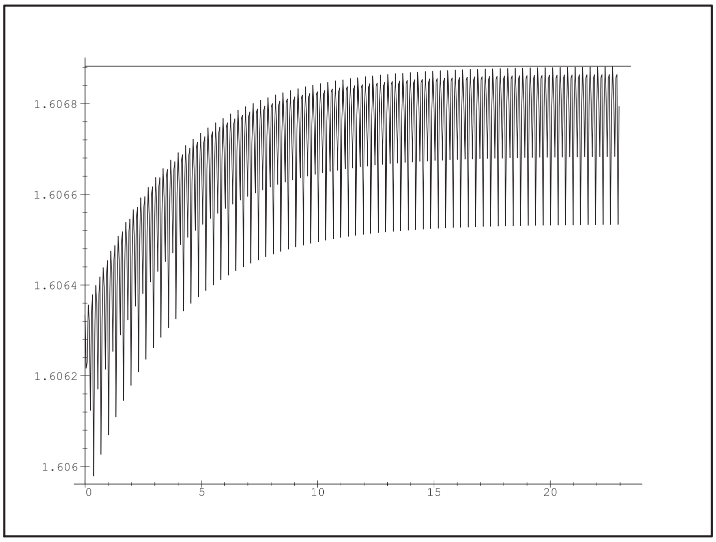

Fig. 1. $Q_{\max}$ and $Q(t_i)$, $i = 0, 1, 2, \ldots, 500$

The following Figure 1 visualizes 500 steps of the computations in the plane $(t, Q(t))$. The horizontal line goes through $Q_{\max}$ and all points $\{(t_i, Q(t_i))\}$, $i = 1, 2, \ldots, 500$, are connected by lines.

REFERENCES

[1] F. Clarke, Yu. Ledyaev, E. Sontag, A. Subotin. Asymptotic Controllability Implies Feedback Stabilization. *IEEE Transaction on Automatic Control*, **42** (1997), 1394-1407.

[2] N. Dimitrova, P. Zlateva. Study of the Steady-States of Methane Fermentation under Uncertain Data. Lecture Notes on Biomathematics and Bioinformatics'95, (ed. M. Candev), D ATECS Publ., Sofia, 1995, 90-99.
[3] D. Dochain, G. Bastin. Adaptive Identification and Control Algorithm for Nonlinear Bacterial Growth Systems. *Automatica*, **20**, *5* (1984), 621-634.
[4] N. Krasovskii, A. Subbotin. Positional Differential Games. Nauka, Moscow, 1974.
[5] M. Quincampoix, N. Seube. Stabilization of Uncertain Control Systems through Piecewise Constant Feedback. *J. of Math. Analysis and Applications*, **218** (1998), 240-255.
[6] I. Simeonov. Modelling and Control of Anaerobic Digestion of Organic Waste. *Chem. Biochem. Eng. Q.* **8 (2)** (1994), 45-52.

Bulgarian Academy of Sciences
Institute for Mathematics and Computer Science
Section "Biomathematics"
Acad. G. Bonchev str., Block 8, BG-1113 Sofia, Bulgaria
E-mail: krast@bas.bg, nelid@iph.bio.bas.bg

ВЪРХУ КОНСТРУИРАНЕТО НА СТАБИЛИЗИРАЩА ОБРАТНА ВРЪЗКА ЗА ЕДИН ПРОЦЕС НА МЕТАНОВА ФЕРМЕНТАЦИЯ

Михаил Иванов Кръстанов, Нели Стоянова Димитрова

Изследван е един модел на метанов ферментационен процес, описан чрез двумерна управляема система. Пресметната е статична оптимална точка по отношение на даден критерий и е конструрирана обратна връзка, стабилизираща процеса в околност на тази статична точка. Представени са също и числени резултати от системата за компютърна алгебра Maple.