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SPECIAL TRIANGLES AND COMPLEX TRIANGLE COORDINATES

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In the paper, the properties of equilateral, isosceles and right triangles are studied by the use of complex analytic formalism recently introduced by June Lester.

Complex numbers are one among traditional tools for study of the Euclidean plane. There are many books considering the application of complex number in geometry (see [4], [5]) and [9]. Recent progress in this area is due to June Lester. Her triangle series develops the Euclidean plane geometry by the use of complex analytic formalism based on cross ratios (see [1], [2], [3]). Notions of shapes and complex triangle coordinates as a part of this formalism play a main role for generalization of many famous theorems in plane geometry.

In this paper, we apply both complex triangle coordinates and shapes for study of some properties of equilateral, isosceles and right triangles. The following preliminarities are known from [2]. We shall recall the basic assumptions, notations and definitions for complex triangle coordinates. This permits us to use directly the theorems from [2].

Let the Euclidean plane be identified with the complex numbers $\mathbb{C}$ and let $\mathbb{C}_\infty = \mathbb{C} \cup \infty$. The cross ratio of four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{C}_\infty$ (at least two of them are distinct) is the number

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}] = \frac{(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{d})}{(\mathbf{a}-\mathbf{d})(\mathbf{b}-\mathbf{c})}.$$

Three distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{C}$ are collinear if and only if the cross ratio $[\infty,\mathbf{a};\mathbf{b},\mathbf{c}]$ is real. The shape of any triangle $\triangle\mathbf{abc}$ is the number

$$\triangle_{\mathbf{abc}} = [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}].$$

Then, ratios of side lengths and angles of triangle can be expressed by its shape, i.e. $|\mathbf{a}-\mathbf{c}| : |\mathbf{a}-\mathbf{b}| = |\triangle_{\mathbf{abc}}|, \sphericalangle\, \mathbf{bac} = arg(\triangle_{\mathbf{abc}})$. Two triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{pqr}$ are similar or antisimilar when $\triangle_{\mathbf{abc}} = \triangle\mathbf{pqr}$ or $\triangle_{\mathbf{abc}} = \overline{\triangle\mathbf{pqr}}$, respectively.

Let $\triangle\mathbf{abc}$ be a fix non-degenerate triangle with the shape $\triangle = \triangle_{\mathbf{abc}} \in \mathbb{C} \setminus \mathbb{R}$. Then, the complex triangle coordinate of any point $\mathbf{z} \in \mathbb{C}_\infty$ with respect to the base triangle $\triangle\mathbf{abc}$ is the number

$$\mathbf{z}_\triangle = [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\infty, \mathbf{z}; \mathbf{c}, \mathbf{b}][\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}].$$

It is clear , that $\mathbf{z}_\triangle \in \mathbb{C}_\infty$, $\mathbf{a}_\triangle = 1$, $\mathbf{b}_\triangle = 0$ and $\mathbf{c}_\triangle = \infty$. Let $\triangle' = [\infty, \mathbf{b}; \mathbf{c}, \mathbf{a}] = \dfrac{1}{1-\triangle}$ and $\triangle'' = [\infty, \mathbf{c}; \mathbf{a}, \mathbf{b}] = 1 - \dfrac{1}{\triangle}$. Then $\mathbf{z}_{\triangle'} = [\mathbf{z}, \mathbf{b}, \mathbf{c}, \mathbf{a}] = \dfrac{1}{1-\mathbf{z}_\triangle}$, $\mathbf{z}_{\triangle''} = [\mathbf{z}, \mathbf{c}, \mathbf{a}, \mathbf{b}] = 1 - \dfrac{1}{\mathbf{z}_\triangle}$ and $\triangle\triangle'\triangle'' = \mathbf{z}_\triangle \mathbf{z}_{\triangle'} \mathbf{z}_{\triangle''} = -1$. The coordinate map $\mathbf{z} \to \mathbf{z}_\triangle$ preserves the cross ratios. From here, four distinct points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in \mathbb{C}$ are collinear or concyclic if and only if the cross ratio $[\mathbf{p}, \mathbf{q}; \mathbf{r}, \mathbf{s}] = [\mathbf{p}_\triangle, \mathbf{q}_\triangle; \mathbf{r}_\triangle, \mathbf{s}_\triangle]$ is real.

Now, we examine three characteristic properties of equlateral triangles in terms of shapes and complex triangle coordinates. Note that the shape of any equilateral triangle is either $w = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ or $\overline{w} = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$. If $\triangle = \alpha + \beta i$ is the shape of the triangle $\triangle\mathbf{abc}$, then $\triangle\mathbf{abc}$ is right-angled at $\mathbf{a}$ when $\alpha = 0$, $\triangle\mathbf{abc}$ is right-angled at $\mathbf{b}$ when $\alpha = 1$, and $\triangle\mathbf{abc}$ is right-angled at $\mathbf{c}$ when $\alpha^2 + \beta^2 - \alpha = 0$.

Proposition 1. *Let points* $\mathbf{d}$ *and* $\mathbf{e}$ *lie on the sides* $\mathbf{ac}$ *and* $\mathbf{ab}$ *of the triangle* $\triangle\mathbf{abc}$ *and devide them in the same ratio* $2:1$*. Let* $\mathbf{o}$ *be a common point of the lines* $\mathbf{bd}$ *and* $\mathbf{ce}$*. Then,a necessary and sufficient condition for* $|\mathbf{a}-\mathbf{b}| = |\mathbf{b}-\mathbf{c}| = |\mathbf{c}-\mathbf{a}|$ *is* $|\mathbf{o}-\mathbf{a}| : |\mathbf{o}-\mathbf{c}| = \sqrt{3} : 2$ *and* $\sphericalangle\, \mathbf{aoc} = \dfrac{\pi}{2}$.

Proof. Let $\triangle = \triangle_{\mathbf{abc}} = [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}] \in \mathbb{C} \setminus \mathbb{R}$. Then $\mathbf{a}_\triangle = 1, \mathbf{b}_\triangle = 0, \mathbf{c}_\triangle = \infty$. From Theorem 2.1 in [2], it follows that $\mathbf{d}_{\triangle'} = [\infty, \mathbf{d}; \mathbf{a}, \mathbf{c}][\infty, \mathbf{b}; \mathbf{c}, \mathbf{a}] = -2\triangle'$. But $\mathbf{d}_{\triangle'} = \dfrac{1}{1-\mathbf{d}_\triangle}$ and $\triangle' = \dfrac{1}{1-\triangle}$. Thus, we obtain $\mathbf{d}_\triangle = \dfrac{1}{2}(3-\triangle)$. Similarly,$1 - \dfrac{1}{\mathbf{e}_\triangle} = \mathbf{e}_{\triangle''} = [\infty, \mathbf{e}; \mathbf{b}, \mathbf{a}][\infty, \mathbf{c}; \mathbf{a}, \mathbf{b}] = -2\triangle'' = -2(1 - \dfrac{1}{\triangle})$. Hence $\mathbf{e}_\triangle = \dfrac{\triangle}{3\triangle - 2}$. By Complex Ceva Theorem (see [2]), we have $\mathbf{o}_\triangle = \mathbf{d}_\triangle . \mathbf{e}_\triangle = \dfrac{1}{2}\triangle(3-\triangle)(3\triangle-2)^{-1}$. Let us calculate the shape of the triangle $\triangle\mathbf{oca}$.

$$\begin{aligned} \triangle_{\mathbf{oca}} &= [\infty, \mathbf{o}; \mathbf{c}, \mathbf{a}] = [\triangle, \mathbf{o}_\triangle; \mathbf{c}_\triangle, \mathbf{a}_\triangle] = [\triangle, \mathbf{o}_\triangle; \infty, 1] \\ &= [1, \infty; \mathbf{o}_\triangle, \triangle] = [\infty, 1; \triangle, \mathbf{o}_\triangle] = \frac{1-\mathbf{o}_\triangle}{1-\triangle} = \frac{\triangle+4}{4-6\triangle}. \end{aligned} \tag{1}$$

First, we prove the necessary condition. If $\triangle\mathbf{abc}$ is equilateral (see Figure 1), then $\triangle = w$ and $\triangle_{\mathbf{oca}} = (w+i)(4-6w)^{-1} = \dfrac{\sqrt{3}}{2}i$. Hence $|\mathbf{o}-\mathbf{a}| : |\mathbf{o}-\mathbf{c}| = |\triangle_{\mathbf{oca}}| = \sqrt{3} : 2$ and $\sphericalangle\, \mathbf{coa} = arg(\triangle_{\mathbf{oca}}) = \dfrac{1}{2}\pi$. Conversaly, if $|\mathbf{o}-\mathbf{a}| : |\mathbf{o}-\mathbf{c}| = \sqrt{3} : 2$ and $\sphericalangle\, \mathbf{coa} = \dfrac{1}{2}\pi$, then $\triangle_{\mathbf{oca}} = (\triangle+4)(4-6\triangle)^{-1} = \dfrac{\sqrt{3}}{2}i$. From here, $\triangle = w$ i.e. $\triangle\mathbf{abc}$ is equilateral. The sufficient condition is proved.

Proposition 2. *Any side of the triangle* $\triangle\mathbf{abc}$ *is divided to three equal segments by the points* $\mathbf{p}, \mathbf{q} \in \mathbf{bc}$*,* $\mathbf{r}, \mathbf{s} \in \mathbf{ca}$ *and* $\mathbf{t}, \mathbf{u} \in \mathbf{ab}$*. A necessary and sufficient condition for* $|\mathbf{a}-\mathbf{b}| = |\mathbf{b}-\mathbf{c}| = |\mathbf{c}-\mathbf{a}|$ *is the points* $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}$ *to be concyclic.*

Proof. The necessary condition is obvious (see Figure 2). For the sufficient conditions, we assume that the points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}$ are concyclic and $\triangle = \triangle_{\mathbf{abc}} = [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}] \in \mathbb{C} \setminus \mathbb{R}$. As in the proof of Proposition 1., we obtain $\mathbf{p}_\triangle = -\dfrac{1}{2}\triangle$, $\mathbf{q}_\triangle = -2\triangle$, $\mathbf{r}_{\triangle'} =$

$-\dfrac{1}{2}\triangle'$, $\mathbf{s}_{\triangle'} = -2\triangle'$, $\mathbf{t}_{\triangle''} = -\dfrac{1}{2}\triangle''$, $\mathbf{u}_{\triangle''} = -2\triangle''$. From here, $\mathbf{r}_{\triangle} = 3 - 2\triangle$ and $\mathbf{s}_{\triangle} = \dfrac{3-\triangle}{2}$. Since $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are concyclic, the number $[\mathbf{p}_{\triangle}, \mathbf{q}_{\triangle}; \mathbf{r}_{\triangle}, \mathbf{s}_{\triangle}] = \dfrac{1}{2}(2-\triangle)(\triangle+1)$ is real, i.e. $(2-\triangle)(\triangle+1) = (2-\overline{\triangle})(\overline{\triangle}+1)$. Then, $2+\triangle-\triangle^2 = 2+\overline{\triangle}-\overline{\triangle}^2$ or $\triangle-\overline{\triangle} = (\triangle-\overline{\triangle})(\triangle+\overline{\triangle})$. From $\triangle \notin \mathbb{R}$ and $\triangle-\overline{\triangle} \neq 0$, it follows that $\triangle+\overline{\triangle} = 1$. Similarly, $\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}$ are concyclic and $[\mathbf{r}_{\triangle'}, \mathbf{s}_{\triangle'}; \mathbf{t}_{\triangle'}, \mathbf{u}_{\triangle'}] = \dfrac{1}{2}(2-\triangle')(\triangle'+1)$ is real. This means that $\triangle' + \overline{\triangle'} = 1$ or $\dfrac{1}{1-\triangle} + \dfrac{1}{1-\overline{\triangle}} = 1$. The last equality is equivalent to $\triangle.\overline{\triangle} = 1$. From $\triangle+\overline{\triangle} = \triangle.\overline{\triangle} = 1$, $\triangle$ is equal to either w or $\overline{w}$, i.e. $\triangle\mathbf{abc}$ is equilateral.

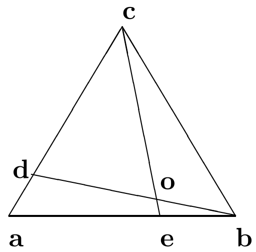

Figure 1

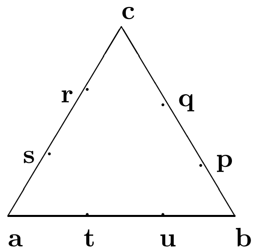

Figure 2

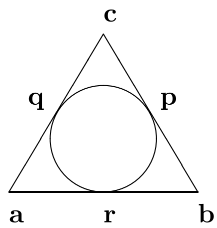

Figure 3

Proposition 3. *The circle inscribed in the triangle* $\triangle\mathbf{abc}$ *touches the sides* $\mathbf{bc}$, $\mathbf{ca}$ *and* $\mathbf{ab}$ *at points* $\mathbf{p}$, $\mathbf{q}$ *and* $\mathbf{r}$, *respectively. A necessary and sufficient condition for* $|\mathbf{a}-\mathbf{b}| = |\mathbf{b}-\mathbf{c}| = |\mathbf{c}-\mathbf{a}|$ *is a similarity of the triangles* $\triangle\mathbf{abc}$ *and* $\triangle\mathbf{pqr}$.

Proof. Let $\triangle\mathbf{pqr}$ be a base triangle and $\triangle = \triangle_{\mathbf{pqr}}$(see Figure 3). Let $\mathbf{s}, \mathbf{t}$ and $\mathbf{u}$ be the midpoints of the segments $\mathbf{qr}, \mathbf{rp}$ and $\mathbf{pq}$, respectively.Then $\mathbf{p}_{\triangle} = 1$, $\mathbf{q}_{\triangle} = 0$, $\mathbf{r}_{\triangle} = \infty$, $\mathbf{s}_{\triangle} = -\triangle$, $\mathbf{t}_{\triangle} = 2-\triangle$, $\mathbf{u}_{\triangle} = \triangle(2\triangle-1)^{-1}$. We observe that the images of $\mathbf{s}, \mathbf{t}, \mathbf{u}$ under the inversion in the inscribed circle are the vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of the triangle. Using Theorem 2.3 [2], we have $\mathbf{a}_{\triangle} = -\overline{\triangle}$, $\mathbf{b}_{\triangle} = 2-\overline{\triangle}$, $\mathbf{c}_{\triangle} = \overline{\triangle}(2\overline{\triangle}-1)^{-1}$. Then,

$$\triangle_{\mathbf{abc}} = [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\triangle, \mathbf{a}_{\triangle}; \mathbf{b}_{\triangle}, \mathbf{c}_{\triangle}] = \frac{\triangle+\overline{\triangle}-2}{2\triangle\overline{\triangle}-(\triangle+\overline{\triangle})}.\overline{\triangle}^2. \tag{2}$$

The triangles $\triangle\mathbf{abc}$ and $\triangle\mathbf{pqr}$ are similar if and only if $\triangle_{\mathbf{abc}} = \triangle$, i.e.

$$\triangle = \frac{\triangle+\overline{\triangle}-2}{2\triangle\overline{\triangle}-(\triangle+\overline{\triangle})}.\overline{\triangle}^2.$$

Set $\triangle = \alpha+\beta i$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\{0\}$. Then, the above equality is equivalent to the system

$$\begin{aligned} \alpha &= (\alpha-1)(\alpha^2-\beta^2)(\alpha^2+\beta^2-\alpha)^{-1} \\ \beta &= -2(\alpha-1).\alpha\beta.(\alpha^2+\beta^2-\alpha)^{-1}. \end{aligned} \tag{3}$$

Since $\triangle\mathbf{pqr}$ is an acute triangle, $\alpha \neq 0, 1$ and $\alpha^2+\beta^2 \neq \alpha$. From here, the system (3) is equivalent to the system $2\alpha\beta^2-\beta^2 = 0 \quad 3\alpha^2-3\alpha+\beta^2 = 0$ with solutions $\alpha = \dfrac{1}{2}, \beta =$

$\pm\frac{\sqrt{3}}{2}$. Hence $\triangle\mathbf{abc}$ and $\triangle\mathbf{pqr}$ are similar, if and only if the shape $\triangle_{\mathbf{abc}} = \triangle_{\mathbf{pqr}}$ is either $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $\overline{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. This completes the proof.

In terms of shapes, there are several ways for determining of an isosceles triangle. We mention three of them. The triangle $\triangle\mathbf{abc}$ with shapes $\triangle = \triangle_{\mathbf{abc}}$ is isosceles with apex at $\mathbf{a}$, i.e. $|\mathbf{a}-\mathbf{b}| = |\mathbf{a}-\mathbf{c}|$ whenever it is fulfilled one of the following equivalent conditions:
(a) $|\triangle| = 1$; (b) $\triangle\overline{\triangle} = 1$; (c) $\frac{1+\triangle}{1-\triangle}$ is pure imaginary.

Proposition 4. *Let* $\mathbf{o}$ *be the circumcentre of the triangle* $\triangle\mathbf{abc}$*. Let* $\mathbf{d}$ *be the midpoint of the side* $\mathbf{ab}$ *and* $\mathbf{e}$ *be the centroid of triangle* $\triangle\mathbf{adc}$*. A necessary and sufficient condition for* $|\mathbf{a}-\mathbf{b}| = |\mathbf{a}-\mathbf{c}|$ *is* $\overrightarrow{\mathbf{eo}} \perp \overrightarrow{\mathbf{dc}}$.

Proof. Let $\triangle\mathbf{abc}$ be a base triangle and $\triangle = \triangle\mathbf{abc}$. Then, $\mathbf{o}_\triangle = \overline{\triangle}$ (see [2], Theorem 2.3). If $\mathbf{m}$ is the midpoint of the side $\mathbf{ca}$, then $\mathbf{e} \in \mathbf{dm}$ and $[\triangle, \mathbf{e}_\triangle; \mathbf{d}_\triangle, \mathbf{m}_\triangle] = [\infty, \mathbf{e}; \mathbf{d}, \mathbf{m}] = -2$(see Figure 4). Using Theorem 2.1 in [2], we have $\mathbf{e}_\triangle = \triangle(5-2\triangle)(4\triangle-1)^{-1}$. From Theorem 3.1 in [2], it follows that $\overrightarrow{\mathbf{eo}} \perp \overrightarrow{\mathbf{dc}}$ if and only if the number

$$\mathcal{R} = \frac{[\triangle, \mathbf{c}_\triangle; \mathbf{e}_\triangle, \mathbf{d}_\triangle]}{[\triangle, \mathbf{e}_\triangle; \mathbf{c}_\triangle, \mathbf{o}_\triangle]} = \frac{3(\triangle-\overline{\triangle})(2\triangle-1)}{5\triangle - 2\triangle^2 + \overline{\triangle} - 4\triangle\overline{\triangle}} \tag{4}$$

is pure imaginary.

If $\triangle\mathbf{abc}$ is isosceles with apex at $\mathbf{a}$, then $\triangle\overline{\triangle} = 1$ and $\frac{1+\triangle}{1-\triangle}$ is pure imaginary. Hence, $\mathcal{R} = \frac{3(\triangle-\triangle^{-1})(2\triangle-1)}{5\triangle-2\triangle^2+\triangle^{-1}-4} = 3\frac{1+\triangle}{1-\triangle}$ is pure imaginary, i.e. $\overrightarrow{\mathbf{eo}} \perp \overrightarrow{\mathbf{dc}}$.

Conversely, if $\overrightarrow{\mathbf{eo}} \perp \overrightarrow{\mathbf{dc}}$, then $\mathcal{R}$ is pure imaginary and $\frac{(2\triangle-1)}{5\triangle-2\triangle^2+\overline{\triangle}-4\triangle\overline{\triangle}}$ is real. This means that

$$\frac{(2\triangle-1)}{5\triangle-2\triangle^2+\overline{\triangle}-4\triangle\overline{\triangle}} = \frac{(2\overline{\triangle}-1)}{5\overline{\triangle}-2\overline{\triangle}^2+\triangle-4\triangle\overline{\triangle}}.$$

From here, it follows that $\triangle\overline{\triangle} = 1$, i.e. $\triangle\mathbf{abc}$ is isosceles with apex at $\mathbf{a}$.

The necessary condition of Proposion 1 as well as Proposions 2 and 3 are known from [7]. The necessary condition of Proposion 4 is a problem from English mathematical competition in 1983 (see [6, p.32].

Proposition 5. *Let* $\mathbf{m}$ *be an interior point of the triangle* $\triangle\mathbf{abc}$*, and let* $\triangle\mathbf{stu}$ *be the pedal triangle of* $\mathbf{m}$ *with respect to* $\triangle\mathbf{abc}$($\mathbf{s} \in \mathbf{bc}, \mathbf{t} \in \mathbf{ca}$ *and* $\mathbf{u} \in \mathbf{ab}$)*. Then,* $\measuredangle\,\mathbf{bmc} = \measuredangle\,\mathbf{bac} + \measuredangle\,\mathbf{tsu}$, $\measuredangle\,\mathbf{cma} = \measuredangle\,\mathbf{cba} + \measuredangle\,\mathbf{uts}$ *and* $\measuredangle\,\mathbf{amb} = \measuredangle\,\mathbf{acb} + \measuredangle\,\mathbf{sut}$.

Proof. Let $\triangle\mathbf{abc}$ be a base triangle and $\triangle = \triangle_{\mathbf{abc}}$. The triangle $\triangle\mathbf{stu}$ is a Miquel triangle of $\mathbf{m}$ to respect $\triangle\mathbf{abc}$(see Figure 5). From Miquel Triangle Shape Theorem (see section 4 in [1]), it follows that

$$\triangle_{\mathbf{stu}} = [\infty, \mathbf{s}; \mathbf{t}, \mathbf{u}] = \overline{[\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}]}.$$

On the other hand $\mathbf{m}_\triangle = [\mathbf{m}, \mathbf{a}; \mathbf{b}, \mathbf{c}] = [\infty, \mathbf{m}; \mathbf{c}, \mathbf{b}]\triangle$. Then, $[\infty, \mathbf{m}; \mathbf{c}, \mathbf{b}].\triangle = \overline{\triangle_{\mathbf{stu}}}$ or $[\infty, \mathbf{m}; \mathbf{b}, \mathbf{c}] = \frac{\triangle}{\overline{\triangle_{\mathbf{stu}}}}$. For $P = \measuredangle\,\mathbf{bmc}$, $A = \measuredangle\,\mathbf{bac}$ and $S = \measuredangle\,\mathbf{tsu}$, we have $|\triangle_{\mathbf{mbc}}|.e^{iP} =$

$\frac{|\triangle|.e^{iA}}{|\triangle_{\mathbf{stu}}|.e^{-iS}}$. Hence, $|\triangle_{\mathbf{mbc}}| = \frac{|\triangle|}{|\triangle_{\mathbf{stu}}|}$ and $P = A + S$. The proofs of the remaining equalities are similar.

Now, we cosider two applications of Proposition 5. The second one is a problem given in mathematical competition (see [8, p.266]).

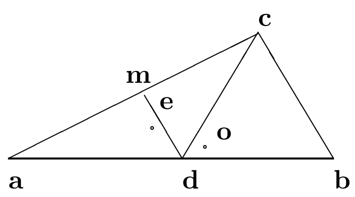

Figure 4

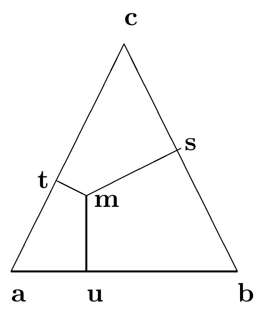

Figure 5

Example 1 *Let* $\triangle\mathbf{abc}$ *be an isosceles and right triangle. Find a point* $\mathbf{m}$ *inside* $\triangle\mathbf{abc}$ *such that the pedal triangle of* $\mathbf{m}$ *with respect to* $\triangle\mathbf{abc}$ *is equilataral.*

Solution. Assume that $\triangle\mathbf{abc}$ is right-angled at $\mathbf{a}$. Let G_1 be the locus of all points $\mathbf{p}$ which satisty $\sphericalangle\,\mathbf{apb} = \frac{7}{12}\pi$, $\mathbf{p}$ lies inside $\triangle\mathbf{abc}$. Similarly, let $G_2 = \{\mathbf{q} : \sphericalangle\,\mathbf{cqa} = \frac{7}{12}\pi,\ \mathbf{q}$ lies inside $\triangle\mathbf{abc}\}$. Then by Proposition 5, $\mathbf{m} = G_1 \cap G_2$.

Example 2. *Let* $\mathbf{m}$ *be a point inside the equilateral triangle* $\triangle\mathbf{abc}$ *and let* $\mathbf{s} \in \mathbf{bc}$, $\mathbf{t} \in \mathbf{ca}$ *and* $\mathbf{u} \in \mathbf{ab}$ *be the feet of the perpendiculars from* $\mathbf{m}$ *to the sides of* $\triangle\mathbf{abc}$. *Find the locus of all points* $\mathbf{m}$ *for which* $\triangle\mathbf{stu}$ *is right-angled.*

Solution. Set $G_1 = \{\mathbf{p} : \sphericalangle\,\mathbf{bpc} = \frac{5}{6}\pi,\ \mathbf{p}$ lies inside $\triangle\mathbf{abc}\}$, $G_2 = \{\mathbf{q} : \sphericalangle\,\mathbf{cqa} = \frac{5}{6}\pi,\ \mathbf{q}$ lies inside $\triangle\mathbf{abc}\}$, and $G_3 = \{\mathbf{r} : \sphericalangle\,\mathbf{arb} = \frac{5}{6}\pi,\ \mathbf{r}$ lies inside $\triangle\mathbf{abc}\}$. It is easy to see that $G_i \bigcap G_j = \emptyset$ for distinct $i, j \in \{1, 2, 3\}$. From Proposition 5., $\triangle\mathbf{stu}$ is right-angled at $\mathbf{s}, \mathbf{t}$ and $\mathbf{u}$, if and only if $\mathbf{m}$ lies on the chord G_1, G_2 and G_3, respectively. Thus, the desired locus is the union $G_1 \bigcup G_2 \bigcup G_3$.

In conclusion, we examine special triangles, but the formulae (1), (2) and (4) occur for an arbitrary triangle.

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СПЕЦИАЛНИ ТРИЪГЪЛНИЦИ И КОМПЛЕКСНИ ТРИЪГЪЛНИ КООРДИНАТИ

Георги Христов Георгиев, Радостина Петрова Енчева, Маргарита Георгиева Спирова

В работата се изучават свойствата на равностранните, равнобедрените и правоъгълните триъгълници чрез използване на комплексно-аналитичния формализъм наскоро въведен от Джун Лестър.