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CALCULATION OF SOME GENERALIZED RAMSEY NUMBERS

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A generalized Ramsey number $R(G1;G2)$ is the minimum n, such that every 2-coloring of the edges of the complete graph Kn contains a monochromatic subgraph isomorphic to G1 or a monochromatic subgraph isomorphic to $G2$.
Consider the graphs $G1 \neq G2$ with no loops or multiply edges and no isolated points with maximum 4 and 5 vertices.
In this paper are proved with details some of the values of the generalized Ramsey numbers with 4 and 5 vertices, which proofs are not accessible or are not known.

A generalized Ramsey number $R(G1;G2)$ is the minimum n, such that every 2-coloring (for example black-white coloring) of the edges of the complete graph Kn contains a monochromatic (black) subgraph isomorphic to $G1$ or a monochromatic (white) subgraph isomorphic to $G2$.

Solid lines are used for black coloring of the edges and dashed lines for white.

The graphs with maximum 4 vertices without any isolated vertex are represented on the Figure 1:

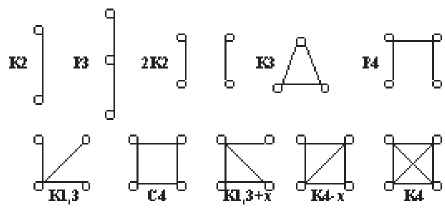

Figure 1.

Theorem 1. $R(K1,3+x;K4-x)=7$*: Every 2-coloring of the edges of the complete graph K7 contains a black subgraph K1,3+x or a white subgraph K4-x. (See Figure 2.)*

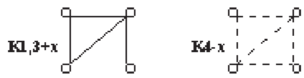

Figure 2.

Proof. We assume that there is a 2-coloring of $K7$ without black $K1,3+x$ and white $K4-x$. Then there is a vertex, which has at least 4 monochromatic edges. In the opposite case, every vertex has to have 3 adjacent black and 3 adjacent white edges. But there is no regular graph with an even number of odd degree vertices.

1). Let the vertex $V1$ be adjacent to black edges $[V1,Vi]$, $i=2,3,4,5$. If any two vertices between $[V2,V3,V4,V5]$ are connected with black edges x, then there is black graph $K1,3+x$. But if the last four vertices are connected with white edges, then there is white $K4-x$.

2). Let the vertex $V1$ be adjacent to white edges $[V1,Vi]$, $i=2,3,4,5$. This induces 2-coloring in subgraph $K4$, created from $V2,V3,V4,V5$ without white $P3$. But $R(P3,C4)=4$ and thus in the subgraph $K4$ there is a black $C4$. Both diagonals of $C4$ are white. See Figure 3.

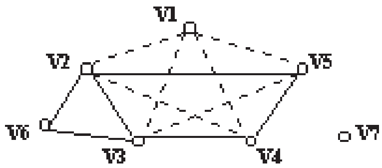

Figure 3

At least one of two edges $[V6,V2]$, $[V6,V4]$ or $[V6,V3]$, $[V6,V5]$ must be black, in which case, $K1,\ 3+x$ must also be black, and thus it is proven that:

$$R(K1,3+x;K4-x)\leq 7. \quad (1)$$

Figure 4 shows 2-coloring of $K6$ without a black $K1,\ 3+x$ or white $K4-x$ and thus:

$$(K1,3+x;K4-x)\geq 7. \quad (2)$$

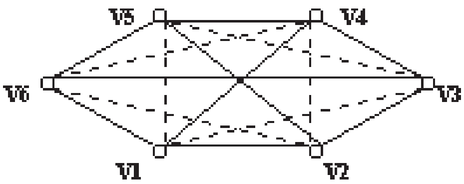

Figure 4.

From (1) and (2) it follows that $R(K1,3+x;K4-x)=7$.

Theorem 2. $R(C4;k4-x)=7$. *Every 2-coloring of the edges of the complete K4 graph contains a black $C4$ subgraph or a white $K4-x$ subgraph. See Figure 5:*

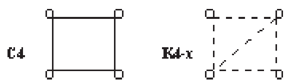

Figure 5

Proof.

(3) $$R(C4; K4 - x) \leq 7.$$

Consider an arbitrary black-white coloring of $K7$, and assume that there is no white $K4 - x$ or black $C4$.

1). Let vertex $V0$ be a white neighbour of $V1, V2, V3$ and $V4$, and let the edge $[V1, V2]$ be black. Vertex $V3$ is a black neighbour with at least one of the vertices being $V1$ or $V2$ (e.g. $[V2, V4]$ is black). In the same way, $V4$ is a black neighbour of $V1$ or $V3$ (e.g. $[V3, V4]$ is black), and then $[V1, V2, V3, V4]$ will be a black $C4$. This is why $[V1, V4]$ must be white. It follows that $[V2, V4]$ and $[V1, V3]$ must be black, and $[V1, V2, V4, V3]$ must be a black $C4$. See Figure 6.

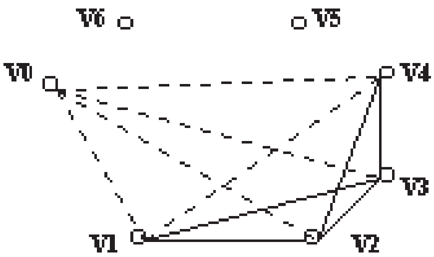

Figure 6

2). Let the vertex $V0$ be a black neighbour of $V1, V2, V3$ and $V4$. At least two vertices among $V1, V2, V3$ and $V4$ are neighbours with a black edge. For example, if the edge $[V1, V2]$ is black, then the edges $[V2, V3]$, $[V2, V4]$, $[V1, V3]$ and $[V1, V4]$ must be white. The vertices $V1, V2, V3$ and $V4$ are adjacent to one black edge and to vertices $V5$ and $V6$. So a black $C4$ is obtained, and thus $R(C4; K4) \leq 7$. See Figure 7.

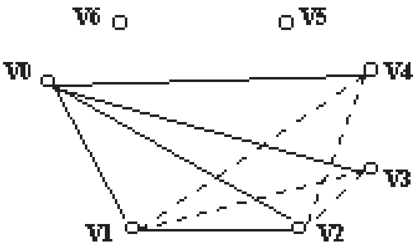

Figure 7

(4) $$R(C4; K4 - x) \geq 7$$

follows from Figure 8 (with only black edges) which represents 2-coloring of $K6$ without a black $C4$ or a white $K4 - x$:

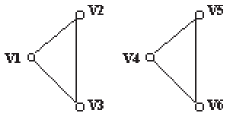

Figure 8

From (3) and (4) it follows that $R(C4; K4 - x) = 7$.

Theorem 3. $R(G7) = 10$*: Every 2-coloring of the edges of the complete K10 graph contains a monochromatic* $G7$ *subgraph. See Figure 9:*

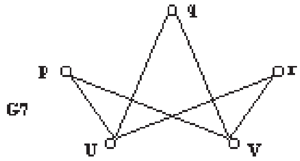

Figure 9

Proof. The inequality

(5) $$R(G7) \geq 10,$$

follows from figure 10, which represents a 2-coloring of $K9$ without a black $G7$ or white $G7$.

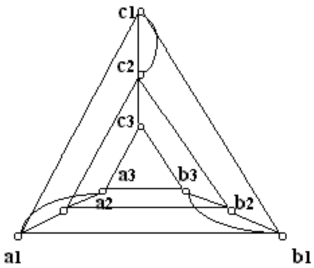

Figure 10

Next, the inequality

(6) $$R(G7) \leq 10$$

will be proven.

Consider the arbitrary black-white coloring of the edges of $K10$. A vertex is considered "black" if its black edges have a degree of at least 5, otherwise it is considered "white". The black neighbours of $V2$ are $U1, U2, U3, U4$ and $U5$. See Figure 11:

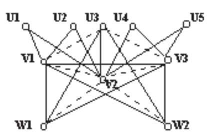

Figure 11

Since there is no black $G7$, vertex $V1$ has a maximum of two black neighbours among the vertices $U1, U2, U3, U4$ and $U5$. The same is true for $V3$.

Therefore, among the vertices Ui ($i = 1, 2, 3, 4$ or 5) there is one (e.g. $U3$), which is a white neighbour of $V1$ and $V3$. There are also two other vertices, $W1$ and $W2$. Vertex $V1$ must be a black neighbour of $W1, W2$, and $V3$. Furthermore, $V1$ must also be a black neighbour of exactly two of the vertices Ui ($i = 1, 2, 3, 4$ or 5) (e.g. $U1$ and $U2$). $V3$ is a black neighbour of $W1, W2, U4, U5$ and $V1$. Vertex $V2$ is a white neighbour of $W1$ and also of $W2$. $U3$ is a black neighbour of $W1$ and also of $W2$. Hence $W1$ and $W2$, together with $V1, U3$ and $V3$, create a black $G7$, which is a contradiction. Thus, $R(G7) \leq 10$.

From (5) and (6) it follows that $R(G7) = 10$.

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ИЗЧИСЛЯВАНЕ НА НЯКОИ ОБОБЩЕНИ ЧИСЛА НА РЕМЗИ

Боряна Дачева Милкоева

Обобщено число на Ремзи $R(G1; G2)$ е минималното естествено число n, такова че при всяко две-оцветяване на ребрата на пълния граф Kn се съдържа или едноцветен подграф, изоморфен на $G1$ или едноцветен подграф, изоморфен на $G2$. Разглеждат се графи $G1 \neq G2$ без примки, без двойни ребра и без изолирани върхове.

В тази статия са доказани подробно някои от стойностите на числата на Ремзи с 4 и 5 върха, чиито доказателства не са достъпни или са неизвестни.