МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2000
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2000
Proceedings of Twenty Ninth Spring Conference of
the Union of Bulgarian Mathematicians
Lovetch, April 3–6, 2000

# THE IMPACT OF COMPUTERS ON TEACHING MATHEMATICS IN SCHOOL

**Rudolf Fritsch**

The latter part of the 20th century can be characterized as a revolutionary time with respect to automatic computing, not only for numerical computations, but also for symbolic algebra allowing for graphical representations of certain results and geometric facts.
Electronic data processing has become ever present and, besides purely mathematical applications, the public schools have to make the students familiar with both the splendour and misery of electronic devices in their different manifestations. In what follows, the main attention is devoted to secondary schools.

**1. Hardware.** Today, not only does every student have his/her own pocket calculator, but many have personal computers at home. There are now classrooms equiped with personal computers, in the style of the special classrooms for physics and chemistry, with the intended aim that not more than two students in the course have to share a PC.

Pocket calculators are used not only in mathematics, but also in such courses as physics, chemistry, and economics. There is a wide variety of pocket calculators on the market. On one end of the scale there are the tiny models, not much larger than a credit or phone card, which have only the basic arithmetic operations. On the other end, one finds the more sophisticated types which are programmable and have fancy software such as the symbolic algebra program DERIVE and the geometry program CABRI already installed. In German schools programmable pocket calculators are not allowed in tests while the others are incorporated in the teaching only from grade 8 onwards. Unfortunately, this rule does not really reflect reality as our children become acquainted with pocket calculators right from kindergarten. Also, the models which are used in school are not of the simplest variety. In addition to the basic arithmetic operations with numbers in decimal representation, they allow for real calculus of fractions, binary, octal and hexadecimal representation of natural numbers

$$500(dec) = 111110100(bin) = 764(oct) = 1F4(hex).$$

Furthermore, they have buttons for square and arbitrary powers, square and cubic roots, exponential functions, logarithms (decimal and natural), trigonometric functions and inverses, mean value, sum of squares and so on. When the mathematics teacher starts teaching with pocket calculators then the tendency is for the whole class to buy the same model.

There is also a certain diversity between schools with respect to personal computers. Although Microsoft has won a very large part of the market, most of our schools now have connection to internet, their home pages and use electronic mail. In most of our schools, a mathematics teacher together with some more or less advanced students, takes care of the homepage and, thus, they form a sort of a corporate identity for the school, at least for these students.

**2. Teachers' training.** At the outset, and before turning to the contents of mathematics teaching with pocket calculators and personal computers, something should be said concerning the preparation of teachers.

Computer science is, up to now, only a voluntary subject in our schools, consequently, it is not a prescribed subject of study within the teachers' education programs. A future teacher for secondary schools has to study two main subjects after which computer science may be added as a third subject. In view of the job market for teachers, such an addition is recommended as it improves the chances for a teaching job in public schools upon completion of the teacher education studies. Since many schools provide the voluntary courses, there is a need for specially trained teachers. In order to meet this demand, special vocational training courses for practicing teachers have been introduced at the Technical University of München, the latter having the strongest computer science department in Bavaria.

In the every day life at school the mathematics teachers, who are often self taught with respect to the computer with almost no specialized training, are taking care of the unavoidable needs of the computer age. While it is often the case that students are more competent than the teachers, there are also computer freaks from other subjects. This results in a high diversity in the school teaching ordered only by few rules in the curricula for mathematics and voluntary computer science.

**3. What is and what should be.**

**Grades 5 to 7.** It has already been stated that pocket calculators are introduced and allowed in the Bavarian schools only from grade 8. If the teachers have already been using the computer classroom, then it is up to them to handle the teaching of very specific topics. The extraction of information from the internet is now widely possible and is also being done in non-mathematical lessons.

From my point of view, grade 8 is already too late. The intention of this rule is that the students can be sufficiently trained in mental and written arithmetic. Clearly, this is unavoidable and there is no way to do this without a mental competence of the multiplication tables up to ten. Also, the students should get a feeling for orders of magnitudes in order to estimate the correctness of a calculator result. But, at the same time, the teaching can not neglect the fact that the students work with pocket calculators and personal computers at home.

Some special topics are worthy of mention in order to stress this idea. In grade 5 the systems for representing numbers are discussed, additive (hieroglyphs, roman numerals) and positional ones (decimal, binary and others). It does not make sense to teach and learn multiplication tables in non-decimal systems, but then the possibilities for actual computations in such systems are very poor. Here the mentioned pocket calculator allows to treat more interesting problems with simple transition from one to the other system, e. g.

$$\begin{aligned} 26 \cdot 19 &= 494\,, \\ 11010 \cdot 10011 &= 111101110\,. \end{aligned}$$

As a handicap, it should be mentioned that according to the restricted number of places in the calculator display, binary computations are only possible up to 1111111111 ($= 511 = 2^9 - 1$).

Another topic in grade 5 consists of some elementary number theory including prime numbers, greatest common divisor and lowest common multiple. Most of students at this age like to play with numbers but find it boring to determine the greatest common divisor only for pairs of numbers such as 36 and 48 or 375 and 225, ie, for numbers whose prime factor decomposition can be found within a reasonable time. The way out of this unsatisfactory situation has already been known for more than 2000 years; it is the Euclidean algorithm. At present, it can and should be used, not only for the computing the greatest common divisor, but also to give the students an initial exposure to the algorithm and computer programs. A PASCAL program for the determination of the greatest common divisor can be easily written down:

```
program euklideanalgorithm;

var
  a,b,q,r : integer;

begin
  write('a = '); readln(a);
  write('b = '); readln(b);
  r:=1;
  repeat
    q:=a div b; r:=a-b*q; a:=b; b:=r;
  until(r=0);
  writeln(a);
end.
```

(Note that the operator “div” gives the integer part (Gauß bracket) of the fraction $a/b$.)

The main topic in grade 6, in German schools, is the calculus of fractions including common and decimal fractions. Although in the everyday life outside the classroom decimal fractions are in use almost everywhere, common fractions are unavoidable for later mathematics instruction. They allow one to describe solutions of equations and the special place of functions more precisely than is possible by the approximative description via finite decimal fractions. In addition, without a thorough treatment of common fractions, rational terms and functions can't be understood. Clearly, at the beginning of the course, there must be a training of the mental skills of the students. But then one should use the pocket calculator means for more advanced computations. There common fractions are presented as mixed numbers as follows:

$$3\frac{1}{4} = 3\lrcorner 1\lrcorner 4\,.$$

This presentation is installed in the pocket calculators used in our classrooms, but it is hard to find in more fancy computer algebra systems like DERIVE, MAPLE or MATHEMATICA.

At the end of grade 6, our students will become acquainted with the geometric tools, ruler, compass and the set square, the modern combination of the ancient protractor and the wooden square. The inaccuracy involved by using these tools is so unsatisfactory that one should restrict oneself to the explanation of the fundamentals and to a discussion of the famous problem of ruler and compass constructions without spending too much time for hand drawings. This mainly concerns the geometric constructions in grade 7 and the following. Look at the students drawings: the circumcircle of a triangle normally comes out ok, but already the incircle is hard to recognize. The same is true for the Euler line, and it is almost impossible to get the touching properties of the Feuerbach circle (nine-point circle). Here the modern geometry softwares like CABRI, EUCLID, THALES and others provide fascinating possibilities. One can explain the idea of construction by ruler and compass:

> *Given a finite set of points, lines and circles in the drawing plane one constructs lines, circles and new points as intersection points of lines and circles by*
>
> - *drawing a line through two given points by means of the ruler and*
> - *drawing a circle with a given point as center passing through another given point by means of the compass.*

In the beginning of the geometry course, one can just imitate these construction steps on the screen with a much higher accuracy than in drawings by hand. But then after having explained the procedure, eg, the perpendicular from a point to a line, the perpendicular bisector or the angular bisector, the system provides a so-called macro which can be seen as a tool to perform this specific construction. That means that the list of tools, ruler and compass, is extended by the named instruments and many more: medians, altitudes of a triangle, circumcircle, incircle and so on. Some macros are loaded with the opening of the system, other can be loaded voluntarily. The latter ones within CABRI are listed in the table 1 (note that the tools provided in the lower part of the table can't be achieved by ruler and compass constructions in the classical sense).

| filename | construction provided |
|---|---|
| 3ehoehen.mac | altitudes and orthocenter of a triangle |
| 3emittls.mac | perpendicular bisectors and circumcenter of a triangle |
| 3eseiteh.mac | medians and centroid of a triangle |
| 3ewnkelh.mac | angular bisectors and incenter of a triangle |
| inkreis.mac | incircle and its center |
| umkreis.mac | circumcircle and its center |
| pentagrm.mac | pentagramm constructed from two vertices |
| skopie.mac | duplication of a line segment |
| wkopie.mac | duplication of an angle |
| 3ewinkel.mac | marking the angles of a triangle |
| ellipse1.mac | ellipse from main axis and one focus |
| ellipse2.mac | ellipse from one axis and a third vertex |
| hyperbel.mac | hyperbola from main axis and a focus |
| parabel.mac | parabola from directrix and focus |
| kstang1.mac | tangent to a conic in a given point of that conic |
| kstang2.mac | tangent to a conic from a given point outside |
| dreitail.mac | angular trisectors |

Table 1. Voluntary macros provided by CABRI

Moreover, the systems allow the definition of a personal macro, for instance, for the excircles of a triangle and it is possible to show the locus of points having a specific property. This opens a new world for teaching plane geometry but ends up with a difficulty of quite another type. The list of suitable theorems and constructions is gigantic, even at the level of grades 7 to 10 in public schools. The big question is what is the right choice for the classroom? I speak in favor of complete freedom for the teachers without a fixed curriculum. This means that the teacher should be free to change his/her choice at any time and, as a bonus,, the teaching becomes a more interesting task.

There is an essential difference in the mathematics curricula between Germany and the countries of the former Soviet Union. In our schools, the trigonometric functions are introduced only in grade 10, while, in the latter, one starts studying these functions in grade 7. Since trigonometric tables are out of print, any reasonable treatment of basic trigonometry requires a pocket calculator or personal computer. It should mentioned here that spherical trigonometry, which was a standard topic at the science gymnasia in Germany up to the first part of the 20th century, was eliminated from the curriculum in the fifties. This was partly due to the complexity of the necessary computations but now with such computations now taking only seconds, one can study astronomy at least in additional voluntary courses;

On the algebraic side, I don't see much use for electronic computing in grade 7 at least along the lines of the mathematics curricula in Germany. Topics such as negative numbers along with terms and binomial formulae, have to be explained at the blackboard and to be learned by the students. There is more curious impact of automatic computing in this level. Look at the problem

$$6a : 3a =?$$

Thirty years ago there was no doubt about the solution: **2**. But now, if you type the problem in the pocket calculator, substituting the variable $a$ by five, you get the result: **50**. The difference arises from the unwritten bracketing. Formerly, the expression in question has been interpreted as $(6 \cdot a) : (3 \cdot a)$ with the idea that the unwritten multiplication dot binds stronger than the written division double dot. Automatic computing can not have such a rule and reads $((6 \cdot a) : 3) \cdot a$. The discussion in Bavaria on this problem ended with the omission of terms of this type from schoolbooks and teaching.

**From grade 8 on.** There is nothing more to say with respect to plane geometry. The main algebraic topics are fractional terms and equations as well as systems of two linear equations with two unknowns. Here, the softwares on symbolic algebra open the possibility for a very wide range of extensions. While factorization problems were treated almost only for polynomials of degree less than or equal to 2, much more interesting factorization problems can now be taken into account. Treating Heron's formula for the area of triangle in a straightforward way one ends up with the term

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\,,$$

where students can't see a possibility for factorization. So the teacher uses a dirty trick in order to obtain on the blackboard something like

$$(2bc + b^2 + c^2 - a^2) \cdot (2bc - b^2 - c^2 + a^2),$$

instead of the given expression, which can be easily factored in order to obtain the classical form

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c)\,.$$

This factorization is also obtainable within seconds by a computer algebra system from the first named expression.

In the upper grades, the techniques and results of calculus can be very effectively visualized, not only in personal computers, but also in fancy pocket calculators with the implementation of a computer algebra system. The influence of parameters can be illustrated along with the discussion of families of functions and their envelopes. The possibilities opened by modern systems raised a strong discussion in Germany which is still ongoing. Up to now, argumentation and proofs played an outstanding role in mathematics education - even in public schools. But computers allow the illustration of a lot of phenomena so that students can be visually confronted with more deep results of mathematics and there is not enough time for exact reasoning. I should emphasize that this discussion takes place mostly among the gymnasium teachers who often don't listen very much to what mathematicians and mathematics educators at the universities are saying. On the one hand, there are teachers who insist on the classical form of mathematics instruction, while on the other, one finds the request for dynamical systems and fractal geometry in the classroom. There is a statement from the Bavarian Government concerning this subject.

> Research results from non-linear dynamics and fractal geometry became known to the public mainly by its esthetically fascinating pictures. Furthermore, the widespread interest and the great response are also founded in the expectations which are anticipated from these results in view of a changing philosophy and conception of the world.
>
> According to a decision of the Bavarian parliament of October 14, 1994, the government has been asked to provide the teachers of mathematics and physics at the gymnasia with material for reasonable inclusion of chaos theory and fractal geometry in the classes.

In voluntary courses, many experiments take place and many teachers are convinced about their success. Some doubts can attributed to the fact that the number of students enrolling at the university in pure and applied mathematics goes down dramatically. But that is another discussion.

Let us turn back to geometry. Descriptive geometry invented by Gaspard Monge more than 200 years ago is now replaced by computer graphics. Since this can be done with the same mathematical strength, and much more effectively, there should be no objection to including computer graphics in the curriculum of public schools. As far as I can see, this process is widely advanced in vocational schools where there is a real need for practical application.

Analytic geometry lost a great part of its former position in school to probability theory and statistics. Now, conics are treated only on a very elementary level such as the demonstration of Dandelin's spheres. Hopefully, computer algebra systems can help to use the remaining time for more interesting topics. As an example, it is possible to show how easily the circumcentre of a tetrahedron can be computed if the vertices $a$, $b$, $c$, $d$ are given by their coordinates in the euclidean 3-space:

```
> with(linalg):
> x:=[x1,x2,x3];

> a:=[3,4,5];
```

$$a := [3, 4, 5]]$$

```
> b:=[1,2,3];
```

$$b := [1, 2, 3]$$

```
> c:=[7,8,10];
```

$$c := [7, 8, 10]$$

```
> d:=[1,0,5];
```

$$d := [1, 0, 5]$$

```
> e1l:=dotprod(x,b-a);
```

$$e1l := -2x1 - 2x2 - 2x3$$

```
> e1r:=subs(x1=(a[1]+b[1])/2,x2=(a[2]+b[2])/2,x3=(a[3]+b[3])/2,e1l);
```

$$e1r := -18$$

```
> e2l:=dotprod(x,c-a);
```

$$e2l := x1 + 4x2 + 5x3$$

```
> e2r:=subs(x1=(a[1]+c[1])/2,x2=(a[2]+c[2])/2,x3=(a[3]+c[3])/2,e2l);
```

$$e2r := \frac{163}{2}$$

```
> e3l:=dotprod(x,d-a);
```

$$e3l := -2x1 - 4x2$$

```
> e3r:=subs(x1=(a[1]+d[1])/2,x2=(a[2]+d[2])/2,x3=(a[3]+d[3])/2,e3l);
```

$$ee3r := -12$$

```
> solve({e1l=e1r,e2l=e2r,e3l=e3r},{x1,x2,x3});
```

$$\left\{x3 = \frac{91}{2}, x2 = \frac{85}{2}, x1 = -79\right\}$$

With respect to probability theory and statistics, some curiosity has to be mentioned. While the former tables for logarithms and trigonometric functions completely disappeared, the students now use tables for probabilistic functions. But it is probably only a matter of time before these functions will also be available on pocket calculators. In addition, statistical results are presented in the form of specific diagrams that clearly now are done also in the classroom since the appropriate software is available.

**4. Final remarks.** Both industry and economy schools are requesting that the use of spreadsheets be taught somewhere in the middle grades of public schools. This topic has not yet found a good and secure place in the curriculum so it is treated more or less voluntarily in different grades attached to different classical topics of mathematics instruction. In some local states of Germany, a new subject "Basic Information Technology" has been introduced in the school curriculum where this kind of material has its natural place.

The trials, about 30 years ago with programmed learning from special developed books, were not very successful. Now many authors develop learning software for all levels of mathematical instruction. The future will show if these electronic tools are more accepted and used than the paper issues.

Finally, let me mention a computer application which is used by all students, teachers and the general public, that of word processing. While student papers take a nice form by this tool, the general public is so inundated with papers and books that it is very difficult to select the useful ones. In closing, it must be a goal for the school to provide students with skills for a sensible criticism.

Address of the Author:
Fakultät für Mathematik
und Informatik der LMU
Theresienstraße 39
80333 München, Deutschland
e-mail: `fritsch@rz.mathematik.uni-muenchen.de`

## ВЛИЯНИЕТО НА КОМПЮТРИТЕ ВЪРХУ ОБУЧЕНИЕТО ПО МАТЕМАТИКА

**Рудолф Фрич**

Разглежда се състоянието на прилагане на информационните технологии в обучението по математика в Германия (по-специално в Бавария) и влиянието им върху методиката на преподаване. Обсъждат се възможностите за приложение на различни схеми и програми съобразно спецификата на обучението по алгебра и геометрия. Предложени са теми по класове, особено подходящи според автора за преподаване с използване на компютри и калкулатори. Разгледана е и системата за квалификация и преквалификация на учителите при използване на новите информационни технологии в обучението по математика.