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FEEDBACK CONTROL OF AN ANAEROBIC FERMENTATION PROCESS UNDER UNCERTAIN DATA*

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A model of continuous methane fermentation process, described by a two-dimensional control system and involving uncertainties in the coefficients is studied. We compute the set of optimal static points according to a given criterion and propose a feedback control stabilizing the process around this set. Numerical results are also reported.

1. Introduction. The methane fermentation is an anaerobic biotechnological process for depollution of organic wastes, resulting in biogas production. There is a variety of mathematical models describing this process [1], [8]–[9] since mathematical modelling has recently becomes a powerful tool for better understanding and simulating of fermentation processes.

We consider a simple model of methane fermentation based on two nonlinear ordinary differential equations [1], [4], [6], and a given criterion for biogas production rate:

$$\frac{dx}{dt} = \frac{\mu_{\max} s}{k_{\rm s} + s} x - ux; \tag{1}$$

$$\frac{ds}{dt} = -k_1 \frac{\mu_{\max} s}{k_{\rm s} + s} x + u(s_{\rm in} - s); \tag{2}$$

$$Q = k_2 \frac{\mu_{\max} s}{k_{\rm s} + s} x, \tag{3}$$

where $x = x(t)$ and $s = s(t)$ are state variables,

x	is biomass concentration,
s	is substrate concentration,
u	is dilution rate,
$s_{\rm in}$	is influent substrate concentration,
$\mu_{\max}$	is maximum specific growth rate of microorganisms,
$k_{\rm s}$	is saturation constant,
k_1	is yield coefficient,
k_2	is coefficient,
Q	is methane gas flow rate.

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The control input is the dilution rate u and the output is methane gas flow rate $Q = Q(u)$. For biological reasons, the state variables x, s, the coefficients $\mu_{\max}$, $k_{\rm s}$, k_1 and k_2 as well as u, $s_{\rm in}$ are positive; additionally, $0 < s < s_{\rm in}$ and $u \in \mathcal{U} = \{u|\, 0 \le u < \mu_{\max}\}$ are valid. In [6] we proposed a bounded feedback stabilizing the dynamics (1)–(2) to the so called optimal static point, which is determined according to the criterion $\max_u Q(u)$. It is assumed in [6] that the model parameters $\mu_{\max}$, $k_{\rm s}$, k_1 and k_2 are exactly known.

Practical experiments and some results from parameter estimation show [8] that most of the coefficients in the model (1)–(3) are not exactly known but bounded. Assume now that instead of numerical values for $\mu_{\max}$, $k_{\rm s}$, k_1 and k_2 we are given intervals $[\mu_{\max}]$, $[k_{\rm s}]$, $[k_1]$ and $[k_2]$. The aim of this paper is to compute the set of all optimal static points when the model parameters vary in the corresponding intervals and to construct a bounded feedback control law stabilizing the uncertain control system to this set.

The paper is organized as follows. In Section 2 we describe the steady states of the methane fermentation process involving intervals in the model coefficients. In Section 3 we construct the feedback stabilizing the uncertain control system. In the last section we report some numerical results using the computer algebra system Maple.

2. The Set of Optimal Static Points. The steady states of the process (1)–(3) satisfy the nonlinear system

$$\frac{\mu_{\max} s}{k_{\rm s} + s} - u = 0 \tag{4}$$

$$-k_1 \frac{\mu_{\max} s}{k_{\rm s} + s} x + u(s_{\rm in} - s) = 0. \tag{5}$$

It is shown in [6] that for each u from the admissible interval U,

$$U = \left[0,\ \frac{\mu_{\max} s_{\rm in}}{k_{\rm s} + s_{\rm in}}\right) \subset \mathcal{U}$$

the nonlinear system (4)–(5) possesses a unique positive solution $(s^*(u), x^*(u))$, where

$$s^*(u) = \frac{k_{\rm s} u}{\mu_{\max} - u}, \quad x^*(u) = \frac{s_{\rm in}\mu_{\max} - (k_{\rm s} + s_{\rm in})u}{k_1(\mu_{\max} - u)} \tag{6}$$

and for all $u \in U$ the following equality $s^*(u) + k_1 x^*(u) = s_{\rm in}$ holds true.

By substituting $s = s^*(u)$ and $x = x^*(u)$ in the expression for Q in (3) we obtain the representation

$$Q(u) = \frac{k_2}{k_1} \cdot \frac{-(k_{\rm s} + s_{\rm in})u^2 + \mu_{\max} s_{\rm in} u}{\mu_{\max} - u}.$$

The function $Q(u)$ is called input-output static characteristic of the dynamics (1)–(2). There is a unique point $\hat{u} \in U$ where $Q(u)$ achieves a local maximum, that is $\max_{u \in U} Q(u) = Q(\hat{u})$ and

$$\hat{u} = \mu_{\max}\left(1 - \sqrt{\frac{k_{\rm s}}{k_{\rm s} + s_{\rm in}}}\right).$$

The point $(x^*(\hat{u}), s^*(\hat{u}))$ is called optimal static point of (1)–(3). By substituting $u = \hat{u}$

in the expressions for $s^*(u)$ and $x^*(u)$ from (6) we obtain

$$s^* = s^*(\hat{u}) = \sqrt{k_{\rm s}(k_{\rm s}+s_{\rm in})} - k_{\rm s}, \quad x^* = x^*(\hat{u}) = (s_{\rm in} - s^*)/k_1. \tag{7}$$

Assume that $\mu_{\max} \in [\mu_{\max}] = [\mu_{\max}^-, \mu_{\max}^+]$, $k_{\rm s} \in [k_{\rm s}] = [k_{\rm s}^-, k_{\rm s}^+]$, $k_1 \in [k_1] = [k_1^-, k_1^+]$ and $k_2 \in [k_2] = [k_2^-, k_2^+]$ are valid. Steady states analysis of the process (1)–(3) involving the above intervals in the coefficients is presented in detail in [4]. Here we mention that the admissible interval U becomes

$$\hat{U} = \left[0,\ \frac{\mu_{\max}^- s_{\rm in}}{k_{\rm s}^+ + s_{\rm in}}\right) \subset \mathcal{U}.$$

Consider further s^* and x^* from (7) as functions of $k_{\rm s}$, k_1, defined on $[k_{\rm s}]$, $[k_1]$. Using the monotonicity of $s^*(k_{\rm s})$ we compute the range

$$[s_1, s_2] = \{s^*(k_{\rm s}) |\ k_{\rm s} \in [k_{\rm s}]\},$$

where

$$s_1 = \sqrt{k_{\rm s}^-(k_{\rm s}^- + s_{\rm in})} - k_{\rm s}^-, \quad s_2 = \sqrt{k_{\rm s}^+(k_{\rm s}^+ + s_{\rm in})} - k_{\rm s}^+.$$

The set

$$\mathcal{S} = \left\{(s,x) |\ s_1 \le s \le s_2,\ \frac{s_{\rm in}-s}{k_1^+} \le x \le \frac{s_{\rm in}-s}{k_1^-}\right\} \tag{8}$$

is called optimal static set of the process (1)–(3) involving intervals in the coefficients. This set is visualized in the plane (s,x) by the quadrangle $ABCD$ on Figure 1. One can easily see that the vertices A, B, C and D have coordinates

$$A\left(s_1, \frac{s_{\rm in}-s_1}{k_1^-}\right),\ B\left(s_2, \frac{s_{\rm in}-s_2}{k_1^-}\right),\ C\left(s_2, \frac{s_{\rm in}-s_2}{k_1^+}\right),\ D\left(s_1, \frac{s_{\rm in}-s_1}{k_1^+}\right),$$

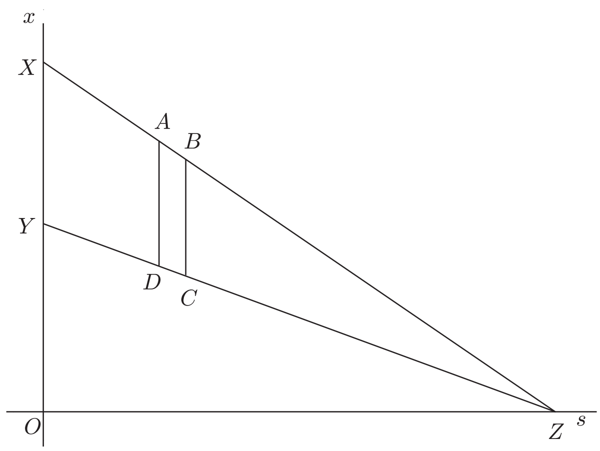

Fig. 1. The optimal static set $ABCD$ and $X(0, \frac{1}{k_1^-})$, $Y(0, \frac{1}{k_1^+})$, $Z(s_{\rm in}, 0)$

and the boundary lines (AB) and (CD) are presented by

$$(AB):\ s + k_1^- x = s_{\rm in}; \qquad (CD):\ s + k_1^+ x = s_{\rm in}.$$

3. Feedback Control Design. Let $d > 0$ and Ω_d be a compact neighbourhood of the optimal static set $\mathcal{S} = ABCD$ consisting of all points $P = (s, x)$ such that $dist_S(P) \leq d$ (here $dist_S$ (P) denotes the distance between the point P and the set $\mathcal{S}$. Following [2], [7] and [10] we shall introduce some notions. A bounded function $k : \Omega_d \to \mathcal{U}$ will be called feedback. Any infinite sequence $\pi = \{t_i\}_{i=0}^{\infty}$ with $0 = t_0 < t_1 < t_2 < \ldots$ and $\lim_{i\to\infty} t_i = \infty$ is called partition of $[0, +\infty)$; the number $d(\pi) = \sup_{i\geq 0}(t_{i+1} - t_i)$ is its diameter. The trajectory associated to a feedback $k(s, x)$ and any given partition π is defined as the solution of (1)–(2) obtained by means of the following procedure (this procedure is borrowed from the theory of positional differential games and is studied in detail in [5]): on every interval $[t_i, t_{i+1}]$ the initial state $(s(t_i), x(t_i))$ is measured, $u_i = k(s(t_i), x(t_i))$ is computed and then the constant control $u \equiv u_i$ is applied until time t_{i+1} is achieved, when a new measurement is taken.

Definition. *The feedback $k : \Omega_d \to \mathcal{U}$ is said to stabilize asymptotically the system* (1)–(2) *to the optimal static set $\mathcal{S}$, if there exist $T > 0$, $\delta > 0$, a partition π with $d(\pi) \leq \delta$ such that for every point $(s, x) \in \Omega_d$ the corresponding trajectory of* (1)–(2) *is well defined on* $[0, +\infty)$ *and satisfies the following conditions:*

(a) $(s(t), x(t)) \in \Omega_d$ *for every* $t \geq 0$*;*

(b) $\lim_{t\to\infty}$ $dist_S(s(t), x(t)) = 0$.

Denote $\tilde{s} = (s_1 + s_2)/2$ and

$$S_u = \{(s,x)|\, k_1^- x + s - s_{\rm in} > 0\}, \quad S_d = \{(s,x)|\, k_1^+ x + s - s_{\rm in} < 0\},$$
$$S_m = \{(s,x)|\, k_1^- x + s - s_{\rm in} \leq 0 \ \wedge \ k_1^+ x + s - s_{\rm in} \geq 0\}.$$

Our main assumption is the following:

Assumption (A). *There exist four positive reals η_d, η_u, η_m and d such that the values of the following functions*

$$u_u^{\pm}(s,x) = \frac{\mu_{\max}^{\pm}(1 + (k_1^{\pm})^2)sx \pm \eta_u(k_s^{\pm} + s)}{(k_s^{\mp} + s)(k_1^-(s_{in} - s) + x)}, \quad (s,x) \in S_u \cap \Omega_d;$$

$$u_d^{\pm}(s,x) = \frac{\mu_{\max}^{\pm}(1 + (k_1^{\pm})^2)sx \pm \eta_d(k_s^{\pm} + s)}{(k_s^{\mp} + s)(k_1^+(s_{in} - s) + x)}, \quad (s,x) \in S_d \cap \Omega_d;$$

$$u_m(s,x) = \begin{cases} 0, & \text{if } s \geq \tilde{s}, \\ \dfrac{k_1^+ \mu_{\max}^+ sx}{(k_s^- + s)(s_{in} - s)} + \dfrac{\eta_m}{(s_{in} - s)}, & \text{if } s < \tilde{s}, \end{cases} \qquad (s,x) \in S_m \cap \Omega_d$$

are admissible values for the control function, i. e. these values of u belong to the compact set $\mathcal{U}$.

Proposition 1. *Let the assumption (A) holds true. Then the control system* (1)–(2)

is asymptotically stabilizable to the optimal static set S by the following feedback

$$k(s,x) \quad = \quad \begin{cases} u_d^-(s,x), & \text{if } (s,x) \in S_d \cap \Omega_d \text{ and } s > \tilde{s}, \\ u_d^+(s,x), & \text{if } (s,x) \in S_d \cap \Omega_d \text{ and } s \le \tilde{s}, \\ u_u^-(s,x), & \text{if } (s,x) \in S_u \cap \Omega_d \text{ and } s > \tilde{s}, \\ u_u^+(s,x), & \text{if } (s,x) \in S_u \cap \Omega_d \text{ and } s \le \tilde{s}, \\ u_m(s,x), & \text{if } (s,x) \in S_m \cap \Omega_d. \end{cases}$$

Proof. The proof is too technical to be given in full length. For that reason we concentrate our selves on some cases which allow us to present its main features. We set $z = (s,x)^T$, $F(z,u) = (f(z,u), g(z,u))^T$, where

$$g(z,u) = g(s,x,u) = \frac{\mu_{\max} s}{k_{\rm s} + s} x - ux, \qquad f(z,u) = f(s,x,u) = -k_1 \frac{\mu_{\max} s}{k_{\rm s} + s} x + u(s_{\rm in} - s).$$

The compactness of the sets Ω_d and $\mathcal{U}$ implies the existence of some real constants $\kappa > 0$, $M > 1$ and $L > 1$ such that for all $z_1,\ z_2,\ z \in \Omega_d$ and $u \in \mathcal{U}$ the following inequalities

$$k(s,x) \ge \kappa, \quad \|F(z,u)\| \le M, \quad \|F(z_1,u) - F(z_2,u)\| \le L\|z_1 - z_2\|$$

hold true. We choose h with

$$0 < h < \frac{1}{2LM} \cdot \min \left\{ \eta_m, \frac{\eta_u}{\sqrt{1 + (k_1^-)^2}}, \frac{\eta_d}{\sqrt{1 + (k_1^+)^2}} \right\}.$$

Let $\pi = \{t_i\}_{i=0}^{\infty}$ with $0 = t_0 < t_1 < t_2 < \dots$ be an arbitrary partition of $[0, +\infty)$ with diameter not greater than h.

Claim 1. Let $z_0 = (s^0, x^0)^T$ be an arbitrary point from the set $\Omega_d \cap S_m$. The trajectory $z(\cdot)$ of (1)–(2) (corresponding to the feedback $k(s,x)$) is well defined on $[0, \infty)$ and there exists a positive real T such that $z(t) \in S$ for $t \ge T$.

Proof of claim 1. We set $\nu^{\pm} = (\pm 1, \pm k_1^{\mp})^T$. It is straightforward to check that

$$\langle \nu^{\pm},\ F(z,u) \rangle \ \le\ 0,\ z \in \Omega_d \cap S_m,\ u \in \mathcal{U}.$$

This inequality implies (cf. e. g. [3], [10]) that every trajectory of the system (1)–(2), starting from a point of S_m remains in S_m. We set $z_i = z(t_i), u_i = k(z_i), i = 0, 1, 2, \dots$. Let us assume that $z_i = (s^i, x^i)^T \in \Omega_d \cap S_m$ and $t \in [t_i, t_{i+1}]$. If $s^i \ge s_2$ then

$$\frac{d}{dt} s(t) = -k_1 \frac{\mu_{\max} s}{k_{\rm s} + s} x \le \eta_l := -k_1^- \frac{\mu_{\max}^- \tilde{s}}{k_{\rm s} + s_{\rm in}} \frac{s_{\rm in} - \tilde{s}}{k_1^+} < 0. \tag{9}$$

Let now $s^i \le s_1$. According to the choice of h we have

$$\begin{aligned} \frac{d}{dt} s(t) \quad &= \quad \frac{d}{dt} s(t_i) + \left(\frac{d}{dt} s(t) - \frac{d}{dt} s(t_i)\right) \\ &= \quad \frac{-k_1 \mu_{\max} s}{k_{\rm s} + s} x + k_1^+ \frac{\mu_{\max}^+ s}{k_{\rm s}^- + s} x + \eta_m + \|F(z(t), u_i) - F(z_i, u_i)\| \end{aligned}$$

$$\geq \eta_m - L\|z(t)-z_i\| \geq \eta_m - L\int_{t_i}^{t} \|F(z(\tau),u_i)\|d\tau$$

$$\geq \eta_m - LM(t_{i+1}-t_i) \geq \eta_m - LMh \geq \frac{\eta_m}{2}.$$

The invariance of the set S_m with respect to trajectories of (1)–(2), the last inequality and (9) imply that $z(t)$, $t \in [t_i, t_{i+1}]$, does not leave the set $S_m \cap \Omega_d$. Therefore, starting with $z_0 \in S_m \cap \Omega_d$ we obtain that $z_i \in S_m \cap \Omega_d$ for all $i = 1, 2, \ldots$. Thus $z(\cdot)$ is defined on $[0, \infty)$. Moreover, the last inequality and (9) imply that $z(t) \in S$ for

$$t > \frac{2\gamma}{\min(\eta_m, \eta_l)}, \quad \text{where } \gamma = \begin{cases} s^0 - s_2, & \text{if } s^0 \geq s_2, \\ s_1 - s^0, & \text{if } s^0 \leq s_1. \end{cases}$$

Claim 2. Let $z_0 = (s^0, x^0)^T$ be an arbitrary point from the set $\Omega_d \cap (S_u \cup S_d)$. The trajectory $z(\cdot)$ of (1)–(2) (corresponding to the feedback $k(s,x)$) is well defined on $[0,\infty)$ and:

(i) there exists a positive real T such that $z(t) \in \Omega_d \cap S_m$ for $t \geq T$;

(ii) $z(t) \in \Omega_d \cap (S_u \cup S_d)$ for every $t \geq 0$ and $\lim_{t\to\infty} dist_S\ (z(t)) = 0$.

Proof of claim 2. Without loss of generality let us assume that $z_0 = (s^0, x^0)^T \in \Omega_d \cap S_u$ and $z(\cdot) = (s(\cdot), x(\cdot))^T$ be the corresponding trajectory. It is straightforward to check that for every $z \in \Omega_d \cap S_u$ and for every $u \in \mathcal{U}$

$$\langle \nu^+, F(z,u)\rangle \leq u \cdot (s_{\text{in}} - s - k_1^- x) = -u \cdot \sqrt{1 + (k_1^-)^2} \cdot dist_{AB}\ (z)$$

$$\leq -\kappa \cdot \sqrt{1 + (k_1^-)^2} \cdot dist_{AB}\ (z)$$

(here $dist_{AB}\ (z)$ denotes the distance between the point z and the line determened by the points A and B).

We set $\tau = (k_1^-, -1)^T$. Clearly, τ is parallel to the line segment AB. As in the proof of Claim 1 we have

$$\langle \tau, \frac{d}{dt} z(t)\rangle \begin{cases} \leq -\eta_u/2 & \text{if } s > \tilde{s}, \\ \geq \eta_u/2 & \text{if } s \leq \tilde{s}. \end{cases}$$

The above estimations for $\langle \nu^+, \frac{d}{dt} z(t)\rangle$ and $\langle \tau, \frac{d}{dt} z(t)\rangle$ show that the trajectory $z(\cdot)$ does not leave the set Ω_d according to Theorem 2.4, p. 191 in [3]. The following two cases are then possible:

(a) there exists a positive real T such that $z(t) \in \Omega_d \cap S_m$ for $t \geq T$;

(b) $z(t) \in \Omega_d \cap S_u$ for every $t \geq 0$.

Let us consider the case (b). The estimations for $\langle \tau, \frac{d}{dt} z(t)\rangle$ show that there exists a positive real T such that $z(t) \in \Omega_d \cap \{(s,x)\ |s_1 \leq s \leq s_2\}$ for all $t \geq T$. We set $\xi(\cdot) = s(\cdot) + k_1^- x(\cdot) - s_{\text{in}}$. Then

$$\frac{d}{dt}\xi(t) = \langle \nu^+, \frac{d}{dt} z(t)\rangle \leq -\kappa \cdot (s_{\text{in}} - s - k_1^- x) = -\kappa \xi(t)$$

and hence, $\xi(t) \leq \exp(-\kappa t) \cdot \xi(0)$. Since $\xi(t) = \sqrt{1 + (k_1^-)^2} \cdot dist_S\ (z(t))$ for $t \geq T$, the last estimation shows that $\lim_{t\to\infty} dist_S\ (z(t)) = 0$ and the proof of Claim 2 is completed.

The proof of Proposition 1 follows directly from Claim 1 and Claim 2. □

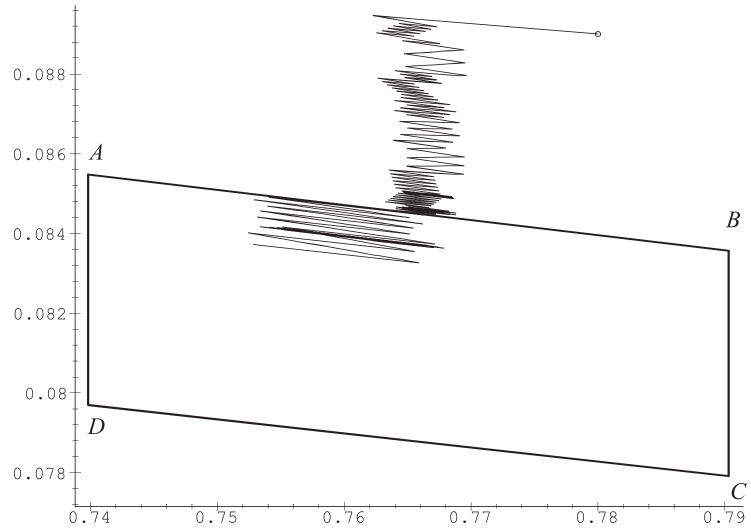

Fig. 2. Feedback control with initial point $(0.78, 0.089)$

4. Numerical Experiments. For our computer simulation we take the following average values for the coefficients in the model (1)–(3), see e. g. [1], [4], [6]:

$$\mu_{\max} = 0.4; \quad k_{\mathrm{s}} = 0.4; \quad k_1 = 27.4; \quad k_2 = 75; \quad s_{\mathrm{in}} = 3.$$

We consider the above values for $\mu_{\max}$, k_{s}, k_1 and k_2 as centers of the corresponding intervals; the radii are given by $r_\alpha \cdot \alpha$, $\alpha \in \{\mu_{\max}, k_{\mathrm{s}}, k_1, k_2\}$ with $0 < r_\alpha < 1$.

All computations presented below are performed by the computer algebra system Maple V Release 3. The designed worksheet proceeds as follows. We start with some initial values $s(0)$, $x(0)$ and an appropriate control according to Proposition 1. With randomly chosen points for $\mu_{\max}$, k_{s}, k_1, k_2 from the corresponding intervals we solve numerically the system (1)–(2) on a mesh $t_i = (i-1)h$, $i = 1, 2, \ldots, n$; thereby at any point t_i we pick out the appropriate feedback. After n steps we choose new random values for the coefficients and repeat the process.

With $r_{k_1} = 0.035$ and $r_\alpha = 0.1$ for $\alpha \in \{\mu_{\max}, k_{\mathrm{s}}, k_2\}$ Figure 2 visualizes the numerical outputs for initial data $s(0) = 0.78$, $x(0) = 0.089$.

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ОБРАТНА ВРЪЗКА ЗА УПРАВЛЕНИЕ НА ЕДИН АНАЕРОБЕН ФЕРМЕНТАЦИОНЕН ПРОЦЕС В УСЛОВИЯ НА НЕОПРЕДЕЛЕНОСТ

Михаил Кръстанов, Нели Димитрова

Разгледан е един непрекъснат модел на метанова ферментация, описан чрез двумерна управляема система при наличие на неопределености в коефициентите. Пресметнато е множеството на оптималните статични точки по отношение на даден производствен критерий. Конструирана е стабилизираща обратна връзка. Представени са и резултати от числени експерименти извършени с помощта на системата за компютърна алгебра Maple.