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DISTRIBUTION OF THE CRITICAL POINTS OF WEIERSTRASS PROCEDURE *

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In this paper, the distribution of the critical initial approximations of Weierstrass' method is considered. Numerical examples, which illustrate our results are given, too.

1. Introduction. Investigations of divergent starting points for every numerical method for finding all roots of a given polynomial show that for any monic polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \tag{1}$$

of degree n, there exists a set $G_f \subset C^n$ such these methods, starting from $\mathbf{z_0} = \mathbf{z} \in G_f$, do not converge to the roots of P. This NS – set [4] is obtained as the set of solutions of a **nonlinear systems** of n equations. Moreover, in general, sets of NS - type are not the only divergent ones. The study of the distribution of the critical initial points is very compicated.

2. Main results. One of the most efficient methods for the simultaneous approximations of all simple polynomial zeros is the Weierstrass method

$$z_i^{k+1} = z_i^k - \frac{P(z_i^k)}{\prod\limits_{j\neq i}^{n}(z_i^k - z_j^k)}, \ (i = 1, \ldots, n;\ k = 0, 1, 2, \ldots). \tag{2}$$

Many authors have observed that method (2) possesses a global convergence in practice for almost all starting points. This statement can be precised. This was proved for $n = 2$ and for cubic polynomial $P(z) = z^3$, but this is an open problem in the general case $n \geq 3$.

The following divergence theorem is more often applicable [3]:

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Let $P(z)$ be a monic polynomial of degree n with $a_0 \neq 0$. If the vector of initial approximations $\mathbf{z^0} = (z_1^0, \dots, z_n^0)$ satisfies the system

$$
\begin{aligned}
&a_{n-1} = 0,\\
&\sum_{l<s}^{n} z_l^0 z_s^0 + a_{n-2} = 0,\\
&2\sum_{l<s<t}^{n} z_l^0 z_s^0 z_t^0 - a_{n-3} = 0,\\
&\dots\\
&(n-1)\prod_{j=1} z_j^0 + (-1)^n a_0 = 0,
\end{aligned}
\tag{3}
$$

then $\mathbf{z_1} = (0, \dots, 0)$ and the Weierstrass method (2) fails.

Let the critical points, obtained by (3) are z_i^0.

We have the following

Theorem. *Let*

$$R = \left(\max_{0\le k\le n-2} \frac{|a_k|}{n-k-1}\right)^{\frac{1}{n}} > 1.$$

All the points z_i^0, $i = 1, \dots, n$ lie in the ring-shaped region

$$\frac{|a_0|(T-1)}{(n-1)\left(T^{n-2}(T+1)(T^2-T+R^n)-2R^n\right)} \le |z| \le \frac{1}{2} + \sqrt{\frac{1}{4} + R^n} = T. \tag{4}$$

Proof. We observe that there is only one NS - divergent starting vector $z^0 = (z_1^0, \dots, z_n^0) \in C^n$ for the method (2), not counting the permutations of the components of z_i^0 and it is given by the set of roots of the algebraic polynomial

$$Q(z) = (z - z_1^0)(z - z_2^0)\dots(z - z_n^0).$$

We see that

$$Q(z) = z^n - \sum_{i=2}^{n} \frac{a_{n-i}}{i-1} z^{n-i}. \tag{5}$$

The proof is obvious, because a simple rearrangiment of the system (3) gives Vieta's formulae for the polynomial $Q(z)$.

The conditions $a_{n-1} = 0$, $R > 1$ imply that $n \ge 2$. Let z be a zero of $Q(z)$ such that $|z| > 1$. According to [6],we have

$$|z|^n \le R^n(1 + |z| + \dots + |z|^{n-2}) = R^n \frac{|z|^{n-1} - 1}{|z| - 1},$$

i.e.

$$\frac{R^n}{|z|-1} \ge \frac{|z|^n}{|z|^{n-1}-1} > \frac{|z|^n}{|z|^{n-1}} = |z|$$

and

$$|z| \le \frac{1}{2} + \sqrt{\frac{1}{4} + R^n}.$$

Next, we prove that $Q(z)$ has no zero in

$$|z| < \frac{|a_0|(T-1)}{(n-1)\left(T^{n-2}(T+1)(T^2-T+R^n)-2R^n\right)}.$$

Let $t(z) = (1-z)Q(z)$. According to [6], we have

$$\begin{aligned} t(z) &= -\tfrac{a_0}{n-1} - \sum_{k=1}^{n-2}\left(\tfrac{a_k}{n-k-1} - \tfrac{a_{k-1}}{n-k}\right) z^k + z^n + a_{n-2}z^{n-1} - z^{n+1} \\ &= -\tfrac{a_0}{n-1} + \lambda(z). \end{aligned}$$

If $T = \frac{1}{2} + \sqrt{\frac{1}{4} + R^n} < 1 + R^{n-2}$, then

$$\begin{aligned} \max_{|z|=T}|\lambda(z)| &\le T^{n+1} + T^n + |a_{n-2}|T^{n-1} + \sum_{k=1}^{n-2}\left|\tfrac{a_k}{n-k-1} - \tfrac{a_{k-1}}{n-k}\right| T^k \\ &= T^{n+1} + T^n + R^nT^{n-1} + 2R^n\left(T + T^2 + \cdots + T^{n-2}\right) \\ &= T^{n+1} + T^n + R^nT^{n-1} + 2R^nT\tfrac{T^{n-2}-1}{T-1} \\ &= \tfrac{T}{T-1}\left(T^{n-2}(T+1)(T^2-T+R^n) - 2R^n\right). \end{aligned}$$

Hence on $|z| \le T$

$$|t(z)| \ge \frac{|a_0|}{n-1} - |\lambda(z)| \ge \frac{|a_0|}{n-1} - \frac{|z|}{T}\max_{|z|=T}|\lambda(z)|$$

by Schwarz's lemma,

$$|t(z)| \ge \frac{|a_0|}{n-1} - \frac{|z|}{T-1}\left(T^{n-2}(T+1)(T^2-T+R^n) - 2R^n\right),$$

and $|t(z)| > 0$ if

$$|z| < \frac{|a_0|(T-1)}{(n-1)\left(T^{n-2}(T+1)(T^2-T+R^n)-2R^n\right)}.$$

This completes the proof of the theorem.

Remark. Denoting by $z_1, \ldots, z_n$, the zeros of the polynomial $Q(z)$ from (5) with z_n being a zero of smallest modulus, we have

$$|z_n|^n \le |z_1 z_2 \ldots z_n| = \frac{|a_0|}{n-1},$$

whence

$$|z_n| \le \left(\frac{|a_0|}{n-1}\right)^{\frac{1}{n}}.$$

Different aspects of this field can be found in [5], [2].

Example 1. For illustration, we consider NS-divergent starting vector of the equation

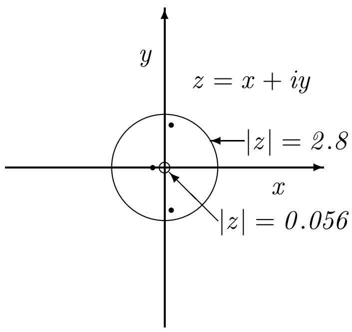
$$P(z) = z^3 + 2z^2 - 5z - 6 = 0.$$

The exact zeros of $P(z)$ are 2, -3, -1. The corresponding NS - polynomial $Q(z)$ of the form (5) is given by

$$Q(z) = z^3 + 5z + 3.$$

The zeros of $Q(z)$ are

$$\begin{aligned}
&z_1 = -0.564099733,\\
&z_2 = 0.282049866 - 2.288811128i,\\
&z_3 = 0.282049866 + 2.288811128i,\\
&|z_2| = |z_3| = 2.30612413.
\end{aligned}$$



All zeros of the polynomial $Q(z)$, i.e. the critical initial approximations for the Weierstrass procedure, lie in the ring-shaped region

$$0.0561 \le |z| \le 2.7919.$$

Example 2. Concerning specific polynomials, we considered polynomials of the following classes

$$P_n(z) = a_0 z^n + a_2 z^{n-2} + a_4 z^{n-4} + \cdots + a_n,$$

where

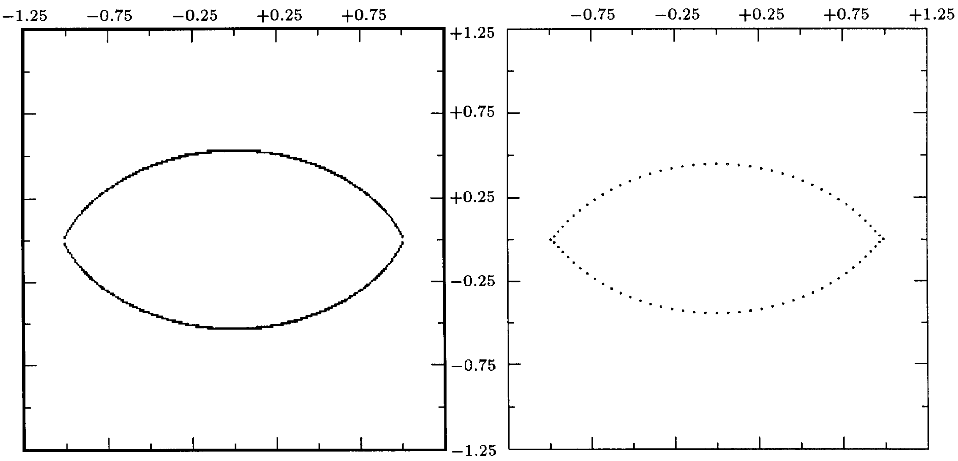
$$\begin{aligned}
&a_0 = 1, a_2 = -\tfrac{n}{6},\\
&a_{2k} = -\tfrac{n}{2k} \sum_{j=1}^{k} \tfrac{1}{2j+1} a_{2(k-j)},\\
&k = 2, 3, \ldots
\end{aligned}$$

The zeros of $P_n(z)$ are the abscissae of Chebyshev numerical integration formulae.

It has been conjuctured by S. Moriguti that all the zeros of

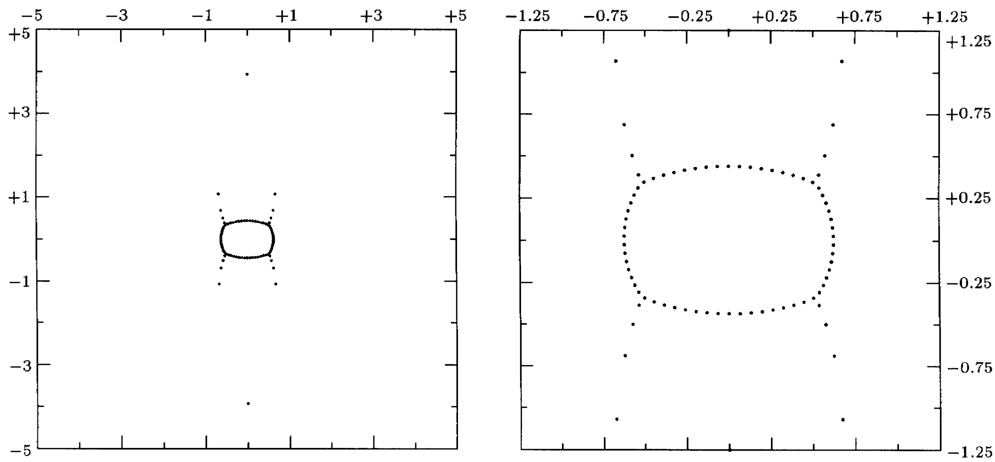
$$P_n(z) = z^n + a_2 z^{n-2} + \cdots + \begin{cases} a_{n-1} z & \text{if n is odd} \\ a_n & \text{if n is even} \end{cases}$$

for $n \to \infty$ would be arranged densely on a closed curve which approaches a closed curve



a) Contour $\left|\dfrac{(z+1)^{\frac{z+1}{2}}}{(z-1)^{\frac{z-1}{2}}}\right| = 2.$ b) Roots of the polynomial $P_{80}(z)$.

Fig. 1. The conjecture by S. Moriguti.



a) Roots of the polynomial $Q_{80}(z)$. b) Roots of the polynomial $Q_{80}(z)$ /zoom/.

Fig. 2. Critical points of the Weierstrass' procedure.

defined by (see Fig. 1)

$$\left|\frac{(z+1)^{\frac{z+1}{2}}}{(z-1)^{\frac{z-1}{2}}}\right| = 2$$

Let $n = 80$. The corresponding NS-polynomial $Q_{80}(z)$ is given by

$$Q_{80}(z) = z^{80} - \sum_{i=1}^{40} \frac{a_{2i}}{2i-1} z^{80-2i}.$$

The non-attractive initial approximations for the iterative method (2) (zeros of Q_{80}) are visualized in Fig.2.

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РАЗПРЕДЕЛЕНИЕ НА КРИТИЧНИТЕ ТОЧКИ НА ИТЕРАЦИЯТА НА ВАЙЕРЩРАС

Николай Веселинов Кюркчиев, Владимир Христов Христов

В тази статия се третират въпроси свързани с разпределението на критичните начални апроксимации за итерацията на Вайерщрас.