МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2000
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2000
Proceedings of Twenty Ninth Spring Conference of
the Union of Bulgarian Mathematicians
Lovetch, April 3–6, 2000

MODELLING OF FINANCIAL MARKETS IN HIGH LEVEL OF INFLATION *

Leda Dimitrova Minkova

The paper describes the basic methodology that can be used in modelling of financial markets. A model which incorporates the domestic price level, the price of foreign zero coupon bond, the stock and the exchange rate is studied. The conditions of absence of arbitrage opportunities are described. The return on domestic risky assets and the change of exchange rate are estimated. The results are related to the presence or absence of currency board conditions.

1. Introduction. Economic crisis are usually accompanied by a high level of inflation. The inflation is a symptom of instability in an economic system. For countries with such type of conditions it is natural to suppose that the foreign economics is in some relatively stable state. In this situation the basic problem for the corresponding National Bank is to determine the main parameters, in order to move the domestic economy to the acceptable stable state.

Platen and Rebolledo [6], modelling the fluctuation of prices of contingent claims, interest rates and exchange rates, describe three basic principles for modelling of financial markets. The first principle is that for any asset and any time the cumulative amount of units sold has to be equal to the cumulative amount of units bought, and trading activity itself depends on the demand and supply, according to the model. The second principle requires conditions, which exclude instantaneous arbitrage opportunities. The third principle imposes minimization of the arbitrage information increase.

In this paper we follow the Platen and Rebolledo's principles. In Section 2 the model is described. Taking into account the second principle and using the methodology of Musiela [5], in Section 3 we estimate the return on risky assets and the exchange rate. In Section 4 we determine the arbitrage information process and we establish the Minimum Arbitrage Information Estimations (MAIE) for the basic characteristics.

2. The model. Let us consider a financial market where investors can trade continuously over the time interval $[0,T]$. Let the stochastic environment of the market be the Brownian motion $W(t) = \{(W_1(t), W_2(t)),\ 0 \le t \le T\}$ in R^2 fixed on the complete probability space $(\Omega, \mathcal{F}, F, P)$, where $F = \{\mathcal{F}_t,\ 0 \le t \le T\}$ is the filtration generated by $\{W(s), 0 \le s \le t\}$ and satisfies the usual conditions. The σ-algebra $\mathcal{F}_t$ represents the information available to the market at time t.

*This research is partially supported by FAPESP grant 98/05810-9.

We assume that the market consists of the domestic price level (or domestic money market account) $P(t)$, the foreign zero-coupon bond $B(t)$, the stock $S(t)$ and the exchange rate $R(t)$. This means that 1 unit of the foreign currency can be exchanged for $R(t)$ units of domestic currency at time t.

Over the time interval $[0, T]$ the assets satisfy the following Ito diffusion equations:

$$dP(t) = P(t)[\pi(t)dt + \sigma_P(t)dW_1(t)], \tag{1}$$

$$dB(t) = B(t)r(t)dt, \quad r(t) > 0, \tag{2}$$

$$dS(t) = S(t)[\mu(t)dt + \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t)] \tag{3}$$

and

$$dR(t) = R(t)[\beta(t)dt + \sigma_R(t)dW_1(t)], \tag{4}$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions. The coefficients are predictable, satisfying regularity conditions to allow existence of strong solutions. If the coefficients are constants there exist unique solutions, see Duffie [3]. If $P(t)$ is the domestic price level, the coefficient $\pi(t)$ in (1) can be interpreted as an inflation coefficient. If $P(t)$ is the domestic money market account or zero-coupon bond, $\pi(t)$ is its return process. In the presence of high level of inflation, zero-coupon bond and money market account are really risky assets. The coefficient $r(t)$ is the return on foreign zero-coupon bond, $\mu(t)$ is the return on domestic risky asset and $\beta(t)$ is the change of exchange rate. The processes $\sigma_P(t), \sigma_1(t), \sigma_2(t)$ and $\sigma_R(t)$ are positive bounded volatility coefficients. We suppose that the stochastic process describing the randomness in the exchange rate is the same as in the price level.

3. Absence of instantaneous arbitrage opportunity. For the model described in Section 2 we would like to find the conditions, which ensure dynamics of an equity market. This leads to the principle of excluding of arbitrage opportunities in the financial economics. Arbitrage-free models fit the market price for risk to current asset prices. We will determine the change of exchange rate and the return on risky assets in a way which ensure the conditions of the second Platen and Rebolledo's principle even in a hight level of inflation. This is possible when the return on the risky assets covers the inflation.

In the market defined above we shall choose as the "numéraire" the process $P(t)$ (see Geman et *al.*, [4]), i.e. the prices of all other assets will be evaluated in units of $P(t)$. In mathematical terms the market is arbitrage-free if there exists a probability measure Q, which is equivalent to the real probability measure P, and such that $\dfrac{B(t)R(t)}{P(t)}$ and $\dfrac{S(t)}{P(t)}$ are martingales under the probability measure Q. The probability measure Q is called a martingale measure. If the market is complete the martingale measure Q is unique. Completeness of the market means that there are as many risky assets as there are independent sources of randomness. If $P(t)$ is considered as a domestic price level, the processes $\bar{B}(t) = \dfrac{B(t)R(t)}{P(t)}$ and $\bar{S}(t) = \dfrac{S(t)}{P(t)}$ can be interpreted as the prices of foreign zero-coupon bond and stock in real units. (see Blenman et *al.*, [2]). If $P(t)$ is the domestic money market account, $\bar{B}(t)$ and $\bar{S}(t)$ are the discounted prices of $B(t)R(t)$

and $S(t)$, with stochastic discount factor.

Note that for each $t \in [0,T]$, the process $B(t)R(t)$ represents the prices of foreign zero-coupon bond in domestic currency. In this way $B(t)R(t)$ can be interpreted as an asset in the domestic economy. This is a new possibility for arbitrage.

Using the Ito's formula we obtain the evolution of the price of the foreign zero-coupon bond in domestic currency determined by

$$d(B(t)R(t)) = B(t)R(t)[(\beta(t)+r(t))dt + \sigma_R(t)dW_1(t)], \quad 0 \le t \le T.$$

The prices of the foreign zero-coupon bond and the stock in real units described by the processes $\bar{B}(t)$ and $\bar{S}(t)$ respectively evolve as follows

$$d\bar{B}(t)=\bar{B}(t)\{[\beta(t)+r(t)-\pi(t)-\sigma_P(t)(\sigma_R(t)-\sigma_P(t))]dt+(\sigma_R(t)-\sigma_P(t))dW_1(t)\} \tag{5}$$

and

$$\begin{aligned} d\bar{S}(t) &= \bar{S}(t)\left[(\mu(t)-\pi(t)-\sigma_P(t)(\sigma_1(t)-\sigma_P(t)))dt\right. \\ &\left.+ \bar{S}(t)(\sigma_1(t)-\sigma_P(t))dW_1(t)+\sigma_2(t)dW_2(t)\right]. \end{aligned} \tag{6}$$

The independence between the Brownian motions $\{W_1(t)\}$ and $\{W_2(t)\}$ implies that the volatility process $\sigma(t)$ of the deflated stock is given by $\sigma^2(t) = (\sigma_1(t)-\sigma_P(t))^2+\sigma_2^2(t)$.

The equation (6) can be rewritten as follows

$$d\bar{S}(t) = \bar{S}(t)[(\mu(t)-\pi(t)-\sigma_P(t)(\sigma_1(t)-\sigma_P(t)))dt + \sigma(t)dM(t)],$$

where

$$\begin{aligned} M(t) &= \int_0^t \frac{(\sigma_1(s)-\sigma_P(s))dW_1(s)+\sigma_2(s)dW_2(s)}{\sqrt{(\sigma_1(s)-\sigma_P(s))^2+\sigma_2^2(s)}} \\ &= \int_0^t \rho(s)dW_1(s) + \sqrt{1-\rho^2(s)}dW_2(s). \end{aligned}$$

In the above expression

$$\rho(t) = \frac{\sigma_1(t)-\sigma_P(t)}{\sigma(t)}$$

is a correlation process with $\rho(t) \in [-1,1]$ for every $t \in [0,T]$.

According to the martingale characterization theorem $\{M(t)\}$ is a Brownian motion process, and

$$E[M(t)W_1(t)] = \int_0^t \rho(s)ds.$$

Remark 1. If $\sigma_1(t) = \sigma_P(t) = 0$ then $\rho(t) = 0$ and $\sigma(t) = \sigma_2(t)$. This is the market described by Bermin [1] i.e. the domestic price level (or money market account) is risk-free.

The processes (5) and (6) can be represented in the following form

$$\begin{pmatrix} d\bar{B}(t) \\ d\bar{S}(t) \end{pmatrix} = \begin{pmatrix} \bar{B}(t) & 0 \\ 0 & \bar{S}(t) \end{pmatrix} \left[\begin{pmatrix} \beta(t)+r(t)-\pi(t)-\sigma_P(t)(\sigma_R(t)-\sigma_P(t)) \\ \mu(t)-\pi(t)-\sigma_P(t)(\sigma_1(t)-\sigma_P(t)) \end{pmatrix} dt \right.$$

$$\left. + \begin{pmatrix} \sigma_R(t)-\sigma_P(t) & 0 \\ \sigma(t)\rho(t) & \sigma(t)\sqrt{1-\rho^2(t)} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}\right].$$

We are interested in the following question: Under which kind of conditions there exists a probability measure Q, equivalent to P, and such that $\bar{B}(t)$ and $\bar{S}(t)$ are martingales with respect to the measure Q. A necessary condition for the existence of an unique probability measure is that the matrix

$$\begin{pmatrix} \sigma_R(t)-\sigma_P(t) & 0 \\ \sigma(t)\rho(t) & \sigma(t)\sqrt{1-\rho^2(t)} \end{pmatrix}$$

is invertible (see Duffie, [3]).

In order to construct the measure Q we shall impose the following assumption.

Assumption. *There exists a bounded predictable process* $H(t) = \{(H_1(t), H_2(t), \quad 0 \le t \le T\}$ *with values in* R^2 *such that for all* $t \in [0,T]$

$$\beta(t) = \pi(t) - r(t) - (H_1(t) - \sigma_P(t))(\sigma_R(t) - \sigma_P(t))$$

and

$$\mu(t) = \pi(t) - [\rho(t)H_1(t) + \sqrt{1-\rho^2(t)}H_2(t)]\sigma(t) + \rho(t)\sigma_P(t)\sigma(t).$$

The latter relations define the process $H(t)$ which is called *a market price for risk.* Now the martingale measure Q can be characterized by the Radon-Nikodym derivative process $\Phi = \{\Phi_{t_0,t} : 0 \le t_0 \le t \le T\}$, where

$$\Phi_{t_0,t} = \exp\{\int_{t_0}^t H^*(s)dW(s) - \frac{1}{2}\int_{t_0}^t |H(s)|^2 ds\}. \tag{7}$$

In (7) $H^*(\cdot)$ means a transposed matrix. For all $A \in \mathcal{F}_T$ we define a function $Q(A)$ by the formula

$$Q(A) = \int_A \Phi_{0,T} dP.$$

Because the process H is bounded, Q is a probability measure on $(\Omega, \mathcal{F})$, equivalent to the measure P. From the Girsanov's theorem it follows that under the measure Q, the process

$$\bar{W}(t) = W(t) - \int_0^t H^*(s)ds, \quad 0 \le t \le T,$$

is a two-dimensional Brownian motion.

A new application of the Girsanov's theorem gives the following result.

Proposition 1. *Let the above assumption be true. Then under the measure* Q, *the process* $(\bar{B}(t), \bar{S}(t))$ *is a two-dimensional martingale.*

The process $(\bar{B}(t), \bar{S}(t))$, relative to the measure Q evolves by the following way

$$\begin{pmatrix} d\bar{B}(t) \\ d\bar{S}(t) \end{pmatrix} = \begin{pmatrix} \bar{B}(t) & 0 \\ 0 & \bar{S}(t) \end{pmatrix} \begin{pmatrix} \sigma_R(t)-\sigma_P(t) & 0 \\ \sigma(t)\rho(t) & \sigma(t)\sqrt{1-\rho(t)^2} \end{pmatrix} \begin{pmatrix} d\bar{W}_1(t) \\ d\bar{W}_2(t) \end{pmatrix},$$

where $\bar{W}_1(t)$ and $\bar{W}_2(t)$ are the independent components of the Q–Brownian motion $\bar{W}(t)$.

The Proposition 1 has a simple interpretation: it determines a simple arbitrage-free approach to the market. In this approach the market price of risk $H(t)$ is taken to be that observed in the market, and the parameters $\beta(t)$ and $\mu(t)$ are estimated relatively to the price of risk.

4. Minimization of arbitrage information. The martingale measure Q, defined in the previous section is an "ideal" measure. The difference between the real and "ideal" measures is defined by the market price of risk $H(t)$. Now we will follow the third Platen's principle of minimization of arbitrage information increase. The problem is to estimate the basic parameters in the economics by the minimization of information. We will call these estimations MAIE (Minimum Arbitrage Information Estimations). In the simple model presented above we will estimate the change of exchange rate $\beta(t)$ and the return on domestic risky asset $\mu(t)$.

Let us define the total information functional of the real probability measure P with respect to the martingale measure Q on the interval $[t_0, t]$ as follows

$$I_{t_0,t}(P,Q) = \frac{1}{2} E^Q(h_t \mid \mathcal{F}_{t_0}), \tag{8}$$

where $0 \leq t_0 \leq t \leq T$, (see Platen et *al.*,[6]). Here $h = \{h_t, t \geq t_0\}$ is the Kullback-Leibler information process, defined by

$$h_t = \Phi_{t_0,t}^{-1} \log \Phi_{t_0,t}^{-1}, \quad t \geq t_0,$$

and E^Q denotes the expectation relative to the martingale measure Q. If the process h_t is not Q-integrable, we set $I_{t_0,t}(P,Q) = \infty$. We will call $I_{t_0,t}(P,Q)$ an *arbitrage information* at time t_0 up to time t.

Substituting $\Phi_{t_0,t}$ from (7) in (8) and using the Girsanov's theorem we obtain

$$I_{t_0,t}(P,Q) = \frac{1}{4} \int_{t_0}^{t} E(|H(s)|^2 \Big| \mathcal{F}_{t_0}) ds, \quad t \geq t_0.$$

According to Platen and Rebolledo [6], the increase of arbitrage information at time $t \geq t_0$ is defined as follows

$$\dot{I}_{t_0,t}(P,Q) = \frac{d}{dt} I_{t_0,t}(P,Q) \Big|_{t=t_0}.$$

For our model, the last relation is equivalent to the following one

$$\dot{I}_{t_0,t}(P,Q) = \frac{1}{4} E(|H(t)|^2 \big| \mathcal{F}_{t_0}) = \frac{1}{4}[H_1^2(t) + H_2^2(t)], \tag{9}$$

where

$$H_1(t) = \frac{\pi(t) - \beta(t) - r(t)}{\sigma_R(t) - \sigma_P(t)} + \sigma_P(t)$$

and

$$H_2(t) = \frac{(\sigma_R(t) - \sigma_1(t)\pi(t) + \sigma(t)\rho(t)(\beta(t) + r(t))}{\sigma(t)\sqrt{1-\rho^2(t)}(\sigma_R(t) - \sigma_P(t))} - \frac{\mu(t)}{\sigma(t)\sqrt{1-\rho^2(t)}}.$$

Now, in order to find the minimal value of the arbitrage information increase, one has to minimize the relation (9).

In order to determine those values of the change of exchange rate β_{MAIE} and the return on risky asset μ_{MAIE} which minimize $\dot{I}_{t_0,t}(P,Q)$ we solve the equations

$$\frac{\partial \dot{I}_{t_0,t}(P,Q)}{\partial \beta(t)} = 0 \quad \text{and} \quad \frac{\partial \dot{I}_{t_0,t}(P,Q)}{\partial \mu(t)} = 0.$$

We can summarize the result in the following proposition.

Proposition 2. *The change of exchange rate and the return on risky asset, which minimize (9) are given by the following expressions*

$$\beta_{MAIE}(t) = \pi(t) - r(t) + \sigma_P(t)(\sigma_R(t) - \sigma_P(t))$$

and

$$\mu_{MAIE}(t) = \pi(t) + \rho(t)\sigma_P(t)\sigma(t),$$

for all $t \geq t_0$.

As it can be seen, even in a high level of inflation, it is possible to ensure some acceptable stable state in the economy, if the return on domestic risky assets μ_{MAIE} covers the inflation.

We demonstrate the method in the simple model, presented above and estimate only two characteristics - the return on risky assets and the change of exchange rate. More general forms can be handled basically in the same way.

Remark 2. Let us suppose that the volatility of exchange rate is $\sigma_R(t) = 0$. This means that the exchange rate evolves as a predictable process. In this case the estimation of the return on domestic risky asset is the same. From Proposition 2 the estimation of the change of exchange rate is determined by

$$\beta_{\mathrm{MAIE}}(t) = \pi(t) - r(t) - \sigma_P^2(t). \tag{10}$$

Let us suppose that a country with unstable economics accepts the so called currency board conditions and the government has success in preserving the conditions according to the first Platen and Rebolledo's principle. As usual in currency board conditions the domestic currency, related to the foreign currency, is "fixed", i.e. $\beta(t) = const$. Even under this strong restriction if the inflation $\pi(t)$ is high, the return on foreign zero-coupon bond $r(t)$ will grow up, according to (10).

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Department of Stochastics and Optimization
Technical University of Sofia
e-mail: leda@vmei.acad.bg

МОДЕЛИРАНЕ НА ФИНАНСОВИ ПАЗАРИ В УСЛОВИЯТА НА ВИСОКА ИНФЛАЦИЯ

Леда Димитрова Минкова

В работата се разглежда методологията, която може да се използва при моделирането на финансовите пазари. Изучава се модел, състоящ се от ценовото равнище, цената на чуждестранните облигации с нулев купон, рискови активи и цената на валутата. Описани са условията при отсъствие на арбитражна възможност. Оценяват се доходността на рисковите активи и промяната в цената на валутата. Оценките са представени и в условията на валутен борд.