МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2000
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2000
Proceedings of Twenty Ninth Spring Conference of the Union of Bulgarian Mathematicians
Lovetch, April 3–6, 2000

SUBCRITICAL BRANCHING PROCESSES WITH NON HOMOGENEOUS IMMIGRATION*

Kosto V. Mitov

Subcritical Galton–Watson branching processes with non–homogeneous, state-dependent immigration is considered. It is obtained the asymptotic behaviour of the first and second factorial moments, when the immigration intensity tends to zero. The limit theorems are also proved.

1. Model and basic equations. Let on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ be given two independent sets of nonnegative, integer valued random variables (r.v.):

a) $X = \{X_n(i), i = 1, 2, \ldots, n = 1, 2, \ldots\}$ – a set of independent, identically distributed r.v. with probability generating function (p.g.f.) $f(s) = \mathbf{E}\{s^{X_i(n)}\} = \sum_{k=0}^{\infty} p_k s^k$, $|s| \leq 1$.

b) $Y = \{Y_n, n = 0, 1, 2, \ldots\}$ – a set of independent r.v. with p.g.f. $g_n(s) = \mathbf{E}\{s^{Y_n}\} = \sum_{k=0}^{\infty} q_k(n) s^k$, $|s| \leq 1$.

We define the process $Z_n, n = 0, 1, 2, \ldots$ as follows

$$Z_0 = Y_0, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n+1}(i) + I_{\{Z_n=0\}} Y_{n+1}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

where it is always assumed that $\sum_{k=1}^{0} * = 0$.

The process Z_n defined by (1.1) is a modification of the classical Galton-Watson branching process, which can be described as follows: It starts with $Y_0 > 0$ particles in the 0-th generation and evolves as a Galton-Watson process up to the moment when $Z_n = 0$. Then in the next generation $n + 1$ $Y_{n+1} > 0$ new particles immigrate, and a new Galton-Watson process starts and so on. If the r.v. $Y_n, n = 0, 1, 2, \ldots$ are non-identically distributed, Z_n is a non-homogeneous Markov chain with the state space $\mathbf{Z}_+$. If $g_n(s) \equiv g(s)$, i.e. the distributions of immigrants are equivalent, we obtain the model, which was introduced and investigated by Foster [4] and Pakes [5]. In this case, Z_n is a homogeneous Markov chain.

Let us denote $P_k(n) = \mathbf{P}\{Z_n = k\}$, $k = 0, 1, 2, \ldots$. Then $H_n(s) = \sum_{k=0}^{\infty} P_k(n) s^n = \mathbf{E}\{s^{Z_n}\}$, $|s| \leq 1$ is the p.g.f. of the number of particles existing in the n-th generation.

*The paper is supported by NFSI-Bulgaria, grant No. MM-704/97

We denote $R_n(s) = 1 - H_n(s)$, $R_n = R_n(0) = \mathbf{P}\{Z_n > 0\}$, and $f_0(s) = s$, $f_1(s) = f(s)$, $f_{n+1}(s) = f(f_n(s))$, $n = 2, 3, \ldots$, are the iterations of the function $f(s)$. Denote also $Q_n(s) = 1 - f_n(s)$, $Q_n = Q_n(0)$. It is well-known that (see eg. [2]), that $f_n(s)$ is the p.g.f. of a Galton-Watson process without immigration, starting with one ancestor. For the factorial moments we will use the following notations: $a = f'(1) = \mathbf{E}\{X_n(i)\}$, $2b = f''(1) = \mathbf{E}\{X_n(i)(X_n(i) - 1)\}$, $m_n = g_n'(1) = \mathbf{E}\{Y_n\}$, $c_n = g_n''(1) = \mathbf{E}\{Y_n(Y_n - 1)\}$, $A_n = H_n'(1) = \mathbf{E}\{Z_n\}$, $B_n = H_n''(1) = \mathbf{E}\{Z_n(Z_n - 1)\}$.

The basic tools for the investigation of the process Z_n are the equations for the p.g.f. obtained in [1]:

$$H_0(s) = g_0(s), \quad H_{n+1}(s) = H_n(f(s)) - (1 - g_n(f(s))H_n(0), \tag{1.2}$$

$$H_{n+1}(s) = g_0(f_{n+1}(s)) - \sum_{k=0}^{n}(1 - g_{n-k}(f_k(s))H_{n-k}(0), \tag{1.3}$$

and the equations for the first and second factorial moments A_n and B_n:

$$A_{n+1} = m_0 a^{n+1} + \sum_{k=0}^{n} P_0(k) m_k a^{n-k}, \tag{1.4}$$

$$B_{n+1} = c_0 a^{2(n+1)} + 2bm_0 \frac{a^{n+1}(a^{n+1} - 1)}{a(a-1)} + \sum_{k=0}^{n} P_0(k) c_k a^{2(n-k)} + 2b \sum_{k=0}^{n} P_0(k) m_k \frac{a^{n-k}(a^{n-k} - 1)}{a(a-1)}, \tag{1.5}$$

which can be obtained by differentiating of (1.3) with respect to s and setting $s = 1$, using also the known results (see [2])

$$f_n'(1) = a^n; \qquad f_n''(1) = 2ba^n(a^n - 1)/(a(a-1)). \tag{1.6}$$

2. Basic conditions and results. To the end of the paper we assume the following conditions:

$$0 < a = f'(1) < 1 \quad 0 < 2b = f''(1) < \infty, \quad (\textit{subcritical case}), \tag{2.1}$$

$$d_1 = \sup_n m_n < \infty, \quad d_2 = \sup_n c_n < \infty, \tag{2.2}$$

$$0 < m_n \to 0, \quad c_n \to 0, \quad n \to \infty. \tag{2.3}$$

The condition that the immigration intensity tends to zero is, in some sense, necessary and sufficient for $\lim_{n\to\infty} \mathbf{P}\{Z_n > 0\} = 0$.

Theorem 2.1. *Let the conditions* (2.1), (2.2) *and* (2.3) *hold. Then*

$$\lim_{n\to\infty} \mathbf{P}\{Z_n > 0\} = 0. \tag{2.4}$$

Theorem 2.2. *Let the conditions* (2.1) *and* (2.2) *hold,* $c_n \to 0$, $n \to \infty$ *and also* (2.4) *is satisfied. Then*

$$\lim_{n\to\infty} \mathbf{E}\{Y_n\} = 0. \tag{2.5}$$

So, the behaviour of a subcritical process with immigration in the state zero, which intensity tends to zero, relates to the behaviour of the classical Galton-Watson process without any immigration. The next theorems give some asymptotic results for Z_n, under the different types of convergence in (2.3).

Theorem 2.3. *Assume* (2.1), (2.2) *and* (2.3). *If also*

$$\lim m_n a^{-n} = M, \; 0 < M < \infty, \quad c_n = O(a^n), \quad n \to \infty, \tag{2.6}$$

then, together with $n \to \infty$:

$$R_n = \mathbf{P}\{Z_n > 0\} \sim MKna^n, \tag{2.7}$$

$$A_n = \mathbf{E}\{Z_n\} \sim Mna^n, \tag{2.8}$$

$$B_n = \mathbf{E}\{Z_n(Z_n - 1)\} \sim 2bMna^n/(a(1-a)), \tag{2.9}$$

and

$$\lim_{n\to\infty} \mathbf{E}\{s^{Z_n} | Z_n > 0\} = F(s). \tag{2.10}$$

Theorem 2.4. *Assume* (2.1), (2.2) *and* (2.3). *If also*

$$\sum_{k=0}^{\infty} m_n a^{-n} = M, \; 0 < M < \infty, \quad c_n = o(a^n), \quad n \to \infty, \tag{2.11}$$

then, together with $n \to \infty$*:*

$$R_n = \mathbf{P}\{Z_n > 0\} \sim (m_0 + P/a)Ka^n, \tag{2.12}$$

$$A_n = \mathbf{E}\{Z_n\} \sim (m_0 + P/a)a^n, \tag{2.13}$$

$$B_n = \mathbf{E}\{Z_n(Z_n - 1)\} \sim 2b(m_0 + P/a)a^n, \tag{2.14}$$

and

$$\lim_{n\to\infty} \mathbf{E}\{s^{Z_n} | Z_n > 0\} = F(s). \tag{2.15}$$

where $P \equiv \sum_{k=0}^{\infty} P_0(k) m_k a^{-k} \in (0, \infty)$.

Remark. The function $F(s)$ is the p.g.f. of the conditional limit distribution of the Galton-Watson process without immigration (see (3.2)).

3. Preliminary results. Under the conditions (2.1) the following well-known results for subcritical Galton-Watson processes hold (see [2]):

$$Q_n \sim Ka^n, \quad n \to \infty, \tag{3.1}$$

where $K \in (0, \infty)$,

$$\lim_{n\to\infty} Q_n(s)/Q_n = 1 - F(s), \quad 0 \le s < 1, \tag{3.2}$$

where the p.g.f. $F(s)$ is the unique solution of the functional equation $1 - F(f(s)) = a(1 - F(s))$ and $F(0) = 0, \;\; F(1) = 1, \;\; F'(1) = K^{-1}$,

$$0 < f_n(0) \le f_n(s) \le 1, \quad s \le f_n(s) \uparrow 1, \;\; n \to \infty, \tag{3.3}$$

uniformly in $0 \le s < 1$.

The p.g.f. $g_n(s), \; n = 0, 1, 2, \ldots$ have the following properties:

For $0 \leq s \leq 1$

$$1 - g_n(s) = m_n(1-s) - (c_n(s)/2)(1-s)^2, \tag{3.4}$$

where

$$0 \leq c_n(s) \leq c_n, \quad c_n(s) \to c_n, \quad s \uparrow 1; \tag{3.5}$$

$$m_n(1-s) - (c_n/2)(1-s)^2 \leq 1 - g_n(s) \leq m_n(1-s). \tag{3.6}$$

The proofs of the above results can be found in [2].

The next lemmas state, for easy references, the well known analytical facts.

Lemma 3.1. *If the sequence $x_n \geq 0, n = 0, 1, 2, \ldots$ converges to $0 \leq x < \infty$, and $\sum_{k=0}^{\infty} y_k = y < \infty$ is the convergent series with positive components, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} x_k y_{n-k} = xy. \tag{3.7}$$

Lemma 3.2. *If $x_n \geq 0, n = 0, 1, 2, \ldots$ and $y_n \geq 0, n = 0, 1, 2, \ldots$ are such that $\lim_{n\to\infty} x_n = x > 0$ and $\lim_{n\to\infty} y_n = y > 0$, then*

$$\sum_{k=0}^{n} x_k y_{n-k} \sim xyn, \quad n \to \infty. \tag{3.8}$$

4. Proofs of the basic results.

Proof of Theorem 2.1. We obtain from (1.3), and (3.4), for $s = 0$,

$$R_{n+1} = m_0 Q_{n+1} - \frac{1}{2} c_0(f_{n+1}(0)) Q_{n+1}^2 \tag{4.1}$$

$$+ \sum_{k=0}^{n} m_k P_0(k) Q_{n-k} - \frac{1}{2} \sum_{k=0}^{n} P_0(k) c_k(f_{n-k}(0)) Q_{n-k}^2.$$

Now, (3.1) gives

$$m_0 Q_{n+1} \sim m_0 K a^n \to 0, \quad n \to \infty. \tag{4.2}$$

Using also (3.3) and (3.5) we have

$$c_0(f_{n+1}(0)) Q_{n+1}^2 \sim c_0 K^2 a^{2n} \to 0, \quad n \to \infty. \tag{4.3}$$

Further, (3.1) yields $\sum_{k=0}^{\infty} Q_k < \infty, \quad \sum_{k=0}^{\infty} Q_k^2 < \infty$. Since $0 \leq P_0(k) \leq 1$ and (2.3), then Lemma 3.1 gives $\sum_{k=0}^{n} m_k P_0(k) Q_{n-k} \to 0, \quad n \to \infty$. Similarly, using also (3.5), we obtain

$$0 \leq \sum_{k=0}^{n} P_0(k) c_k(f_{n-k}(0)) Q_{n-k}^2 \leq \sum_{k=0}^{n} P_0(k) c_k Q_{n-k}^2 \to 0, \quad n \to \infty. \tag{4.4}$$

Finally, (4.1)–(4.4) yield (2.4). The theorem is proved. □

Proof of Theorem 2.2. We use the representation (4.1) again. Under the conditions of the theorem (4.2), (4.3) and (4.4) hold. Since $R_n \to 0,\ n \to \infty$, then from (4.1) we obtain

$$\sum_{k=0}^{n} m_k P_0(k) Q_{n-k} \to 0, \quad n \to \infty. \tag{4.5}$$

Let us assume that $\limsup_{n\to\infty} m_n = m > 0$. Hence, there exists a subsequence m_{n_k} such that $m_{n_k} \to m > 0$, $k \to \infty$. From $P_0(k) = 1 - R_k \to 1$, $k \to \infty$ it follows that $m_{n_k} P_0(n_k) \to m > 0$, $k \to \infty$. Using the last relation and the convergence of the series $\sum\limits_{k=0}^{\infty} Q_k$ we get

$$\sum_{j=0}^{n_k} P_0(j) m_j Q_{n_k - j} \geq \sum_{j=n_0}^{n_k} P_0(j) m_j Q_{n_k-j} \to m \sum_{j=0}^{\infty} Q_{n_j} > 0, \quad k \to \infty.$$

In the last two sums j takes as values only the indexes of the subsequence. Therefore, $\liminf_{k\to\infty} \sum\limits_{j=0}^{n_k} P_0(j) m_j Q_{n_k-j} > 0$, which contradicts to (4.5). The theorem is proved. □

Proof of Theorem 2.3. Let $s \in [0,1)$ be fixed. From (1.3) and (3.4) it follows that

$$R_{n+1}(s) = m_0 Q_{n+1}(s) - \frac{1}{2} c_0(f_{n+1}(s)) Q_{n+1}^2(s) \tag{4.6}$$

$$+ \sum_{k=0}^{n} m_k P_0(k) Q_{n-k}(s) - \frac{1}{2} \sum_{k=0}^{n} P_0(k) c_k(f_{n-k}(s)) Q_{n-k}^2(s).$$

First of all, using (3.2) and (3.1) we obtain

$$m_0 Q_{n+1}(s) \sim m_0 K(1 - F(s)) a^{n+1}, \quad n \to \infty. \tag{4.7}$$

From (3.5), (3.3) we obtain

$$0 \leq c_0(f_{n+1}(s)) Q_{n+1}^2(s) \leq c_0 Q_{n+1}^2 \sim c_0 K^2 a^{2n}, \quad n \to \infty. \tag{4.8}$$

Further from Theorem 2.1 and (2.6) it follows that

$$P_0(n) m_n a^{-n} \to M, \quad n \to \infty. \tag{4.9}$$

Furthermore, (3.2) gives $Q_n(s) a^{-n} \to K(1 - F(s))$, $n \to \infty$. Applying Lemma 3.2 we find that when $n \to \infty$,

$$\sum_{k=0}^{n} m_k P_0(k) Q_{n-k}(s) = a^n \sum_{k=0}^{n} \frac{m_k P_0(k)}{a^k} \frac{Q_{n-k}(s)}{a^{n-k}} \tag{4.10}$$

$$\sim MKn(1 - F(s)) a^n.$$

Again from (3.5), (3.3) and (2.6) it follows that for $k \to \infty$ and $n \geq k$, $0 \leq a^{-k} P_0(k) c_k(f_{n-k}(s)) \leq a^{-k} c_k = O(1)$, $n \geq 0$. Moreover, (3.3) and (3.1) immediately yield $\sum\limits_{k=0}^{\infty} Q_k^2(s) a^{-k} < \infty$. From the last two relations, it is easy to conclude that if $n \to \infty$,

$$0 \leq \sum_{k=0}^{n} P_0(k) c_k(f_{n-k}(s)) Q_{n-k}^2(s) \leq a^n \sum_{k=0}^{n} \frac{P_0(k) c_k}{a^k} \frac{Q_{n-k}^2}{a^{n-k}} = O(a^n). \tag{4.11}$$

Finally, (4.6)–(4.11) yield that for each fixed $s \in [0,1)$, $n \to \infty$,

$$R_n(s) \sim MKn(1 - F(s)) a^n. \tag{4.12}$$

Setting $s = 0$ in (4.12) we prove (2.7). For $|s| \leq 1$ $\mathbf{E}\{s^{Z_n} | Z_n > 0\} = 1 - R_n(s)/R_n$. Now, (2.10) follows from (2.7) and (4.12).

The proof of (2.8) follows by the representation (see (1.4)):

$$A_{n+1} = m_0 a^{n+1} + a^n \sum_{k=0}^{n} \frac{P_0(k) m_k}{a^k}, \tag{4.13}$$

and from (4.9), which yields (see [3], Sect.8.9) $\sum_{k=0}^{n} P_0(k) m_k a^{-k} \sim Mn,\ n \to \infty.$

For the proof of (2.9) we will use (1.5). First of all, it is easy to see, that

$$2bm_0 \frac{a^{n+1}(a^{n+1}-1)}{a(a-1)} \sim \frac{2bm_0 a^{n+1}}{a(1-a)}, \quad n \to \infty. \tag{4.14}$$

We estimate the sum $\sum_{k=0}^{n} P_0(k) c_k a^{2(n-k)}$, using also (2.6),

$$\sum_{k=0}^{n} P_0(k) c_k a^{2(n-k)} = a^n \sum_{k=0}^{n} \frac{P_0(k) c_k}{a^k} a^{n-k} = O(a^n),\ n \to \infty. \tag{4.15}$$

Finally, for the last sum in (1.5) we have the representation

$$\sum_{k=0}^{n} P_0(k) m_k \frac{a^{n-k}(a^{n-k}-1)}{a(a-1)}$$

$$= \frac{a^n}{a(1-a)} \sum_{k=0}^{n} \frac{P_0(k) m_k}{a^k} - \frac{a^n}{a(1-a)} \sum_{k=0}^{n} \frac{P_0(k) m_k}{a^k} a^{n-k} = S_1(n) - S_2(n).$$

For $S_1(n)$, we obtain from (4.9) (see also [3], Sect.8.9), $S_1(n) \sim a^n/(a(1-a))Mn$. For $S_2(n)$, again from (4.9), and Lemma 3.1 we get $S_2(n) \sim a^n/(a(1-a)^2)M,\ n \to \infty$. The last three relations imply

$$2b \sum_{k=0}^{n} P_0(k) m_k \frac{a^{n-k}(a^{n-k}-1)}{a(a-1)} \sim \frac{2bMna^n}{a(1-a)},\ n \to \infty. \tag{4.16}$$

Now, combining (4.14)–(4.16) and (1.5) we prove (2.9). The theorem is proved. □

Proof of Theorem 2.4. The proof is quite similar to the proof of Theorem 2.3, one just uses (2.11) instead of (2.6) and we omit it. □

REFERENCES

[1] K. V. Mitov, V. A. Vatutin, N. M. Yanev. Critical Galton-Watson processes with decreasing state-dependent immigration, *Serdica Bul. math. publ.* **10** (1984), 12-24, (in Russian).

[2] B. A. Sevastyanov. Branching Processes, Mir, Moscow, 1970, (in Russian).

[3] W. Feller. An Introduction to Probability Theory and its Applications, vol. 2, Mir, Moscow, 1984, (in Russian).

[4] J. Foster. A limit theorem for a branching process with state dependent immigration. *Ann. Math. Statist.* **42** (1971), 1773-1776.

[5] A. G. Pakes. A branching process with a state-dependent immigration component. *Adv. Appl. Prob.* **3** (1971), 301-314.

Kosto V. Mitov
G. Kochev str., 8, vh. A, apt. 16
5800 Pleven, Bulgaria

ДОКРИТИЧЕСКИ РАЗКЛОНЯВАЩИ СЕ ПРОЦЕСИ С НЕЕДНОРОДНА ИМИГРАЦИЯ

Косто В. Митов

Разглеждат се докритически процеси на Галтон-Уотсън с нееднородна имиграция в състоянието нула. Получени са асимптотически формули за първите два факториални момента, когато интензивността на имиграцията клони към нула. Доказани са гранични теореми.