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# ON SOME SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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The object of this paper is to obtain sharp results involving coefficient bounds, growth and distortion properties for some classes of analytic and univalent functions in the open unit disk.

**1. Introduction and definitions.** Let $S$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1} \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$E := \{z : z \in C \text{ and } |z| < 1\}.$$

We denote by $(f * g)(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $S$ where $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1}. \tag{1.2}$$

Thus

$$(f * g)(z) := z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1}. \tag{1.3}$$

The $n$-th order Ruscheweyh derivative $D^n f(z)$ of a function $f(z)$ in $S$ is defined by

$$D^n f(z) := \frac{z\left(z^{n-1} f(z)\right)^n}{n!} \tag{1.4}$$

where $n$ is any integer such that $n > -1$. It is easy to see from (1.3) and (1.4) that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \tag{1.5}$$

$$= z + \sum_{k=1}^{\infty} \delta(n,k) a_{k+1} z^{k+1} \tag{1.6}$$

where

$$\delta(n,k) := \binom{n+k}{n},\ k \in N. \tag{1.7}$$

The symbol $D^n f(z)$ was called the $n$-th order Ruscheweyh derivative of $f \in S$ by Al-Amiri [1].

Next, let $T$ denote the subclass of $S$ consisting of analytic and univalent functions which can be expressed in the form

$$f(z) = z - \sum_{k=1}^{\infty} a_{k+1} z^{k+1},\ a_{k+1} \geq 0. \tag{1.8}$$

Let $T_n[A,B,\alpha,\beta]$ denote the class of analytic and univalent functions $f(z)$ belonging to the class $T$ and satisfying the additional condition

$$\left| \frac{z^{-1}D^{n+1}f(z) - 1}{Bz^{-1}D^{n+1}f(z) - [B + (A-B)(1-\alpha)]} \right| < \beta \tag{1.9}$$

$$(z \in E;\ 0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1).$$

The class $T_{-1}[-1,1,\alpha,\beta] = P_0^*(\alpha,\beta)$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1)$ was studied by Srivastava and Owa [2].

The class $T_d[A,B,0,1] = T_d[A,B]$ $(d \geq -1;\ -1 \leq A \leq 1;\ -1 \leq B \leq 0;\ B \leq A)$ was studied by Chen [3].

The class $T_0[-1,1,\alpha,\beta] = P^*(\alpha,\beta)$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1)$ was studied by Gupta and Jain [4].

The class $T_0[-1,\mu,\alpha,\beta] = P^*(\alpha,\beta,\mu)$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ 0 \leq \mu \leq 1)$ was studied by Owa and Aouf [5].

The object of the present paper is to obtain some results involving coefficient bounds, growth and distortion properties for the class $T_n[A,B,\alpha,\beta]$.

## 2. A theorem on coefficient bounds.

**Theorem 1.** *A function $f(z)$ defined by (1.8) is in the class $T_n[A,B,\alpha,\beta]$ if and only if*

$$\sum_{k=1}^{\infty} (1+\beta B)\delta(n+1,k)a_{k+1} \leq (B-A)\beta(1-\alpha). \tag{2.1}$$

*The result is sharp.*

**Proof.** Let inequality (2.1) hold true and let $|z| = 1$. Then we obtain

$$|z^{-1}D^{n+1}f(z) - 1| - \beta \left| Bz^{-1}D^{n+1}f(z) - [B + (A-B)(1-\alpha)] \right| =$$

$$\left| -\sum_{k=1}^{\infty} \delta(n+1,k)a_{k+1}z^k \right| - \beta \left| (B-A)(1-\alpha) - B\sum_{k=1}^{\infty} \delta(n+1,k)a_{k+1}z^k \right| \leq$$

$$\sum_{k=1}^{\infty} \delta(n+1,k)a_{k+1} - \beta \left[ (B-A)(1-\alpha) - B\sum_{k=1}^{\infty} \delta(n+1,k)a_{k+1} \right] =$$

$$\sum_{k=1}^{\infty} (1+\beta B)\delta(n+1,k)a_{k+1} - (B-A)\beta(1-\alpha)) \leq 0$$

by the hypothesis of Theorem 1, $\delta(n,k)$ being defined by (1.7). By the Maximum Modulus Theorem, we have

$$f(z) \in T_n[A,B,\alpha,\beta].$$

In order to prove the converse, we assume that $f(z)$ is defined by (1.8) and is in the class $T_n[A,B,\alpha,\beta]$. Then the condition (1.9) readily yields

$$\left|\frac{z^{-1}D^{n+1}f(z)-1}{Bz^{-1}D^{n+1}f(z)-[B+(A-B))1-\alpha)]}\right| =$$

$$\left|\frac{-\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1}z^k}{(B-A)(1-\alpha)-B\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1}z^k}\right| < \beta \qquad z\in E. \tag{2.2}$$

Since $|R(z)|\le|z|$ for all $z$, we find from (2.2) that

$$R\left\{\frac{\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1}z^k}{(B-A)(1-\alpha)-B\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1}z^k}\right\} < \beta \qquad z\in E. \tag{2.3}$$

Now we choose values of $z$ on the real axis so that $z^{-1}D^{n+1}f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1$ through real values we obtain

$$\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1}\le(B-A)\beta(1-\alpha)-\beta B\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1} \tag{2.4}$$

which gives us the desired assertion (2.1).

The assertion 2.1 of Theorem 1 is sharp. The extremal function being

$$f(z)=z-\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)\delta(n+1,k)}z^{k+1} \qquad k\in N. \tag{2.5}$$

**Corollary.** *Let the function $f(z)$ defined by (1.8) be in the class $T_n[A,B,\alpha,\beta]$. Then*

$$a_{k+1}\le\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)\delta(n+1,k)} \qquad k\in N. \tag{2.6}$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

## 3. Growth and distrotion properties.

**Theorem 2.** *Let the function f(z) defined by (1.8) be in the class $T_n[A,B,\alpha,\beta]$. Then for $|z|=r$ $(0<r<1)$,*

$$r-\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r^2\le|f(z)|\le r+\frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r^2, \tag{3.1}$$

$$1-\frac{2(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r\le|f'(z)|\le 1+\frac{2(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r \tag{3.2}$$

*and*

$$1-\frac{(B-A)\beta(1-\alpha)}{1+\beta B}r\le\left|\frac{D^{n+1}f(z)}{z}\right|\le 1+\frac{(B-A)\beta(1-\alpha)}{1+\beta B}r. \tag{3.3}$$

*Each of these results is sharp.*

**Proof.** Since $f(z) \in T_n[A, B, \alpha, \beta]$, in view of Theorem 1, we have

$$(1+\beta B)\delta(n+1,1)\sum_{k=1}^{\infty} a_{k+1} \le \sum_{k=1}^{\infty}(1+\beta B)\delta(n+1,k)a_{k+1} \le (B-A)\beta(1-\alpha) \tag{3.4}$$

which immediately yields

$$\sum_{k=1}^{\infty} a_{k+1} \le \frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}. \tag{3.5}$$

Consequently, for $|z| = r$ $(0 < r < 1)$, we obtain

$$|f(z)| \ge r - r^2\sum_{k=1}^{\infty} a_{k+1} \ge r - \frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r^2 \tag{3.6}$$

and

$$|f(z)| \le r + r^2\sum_{k=1}^{\infty} a_{k+1} \le r + \frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}r^2 \tag{3.7}$$

which prove the assertion (3.1) of Theorem 2.

Furthermore, it is easily seen from (1.8) that, for $|z| = r$ $(0 < r < 1)$

$$|f'(z)| \ge 1 - r\sum_{k=1}^{\infty}(k+1)a_{k+1}. \tag{3.8}$$

But, in view of Theorem 1, we have

$$\begin{aligned} &\frac{(1+\beta B)\delta(n+1,1)}{2}\sum_{k=1}^{\infty}(k+1)a_{k+1} \le \\ &\le \sum_{k=1}^{\infty}(1+\beta B)\delta(n+1,k)a_{k+1} \le (B-A)\beta(1-\alpha) \end{aligned} \tag{3.9}$$

which readily yields

$$\sum_{k=1}^{\infty}(k+1)a_{k+1} \le \frac{2(B-A)\beta(1-\alpha)}{(1+\beta B)(n+r)}. \tag{3.10}$$

Upon substituting from (3.10) into the second members of (3.8) and (3.9), we obtain the assertion (3.2) of Theorem 2.

Next, by using the second inequality in (3.4) we observe that, for $|z| = r$ $(0 < r < 1)$

$$\left|z^{-1}D^{n+1}f(z)\right| \le 1 + r\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1} \le 1 + \frac{(B-A)\beta(1-\alpha)}{1+\beta B}r \tag{3.11}$$

and

$$\left|z^{-1}D^{n+1}f(z)\right| \ge 1 - r\sum_{k=1}^{\infty}\delta(n+1,k)a_{k+1} \ge 1 - \frac{(B-A)\beta(1-\alpha)}{1+\beta B}r \tag{3.12}$$

which prove the assertion (3.3) of Theorem 2.

Equalities in (3.1), (3.2) and (3.3) of Theorem 2 are attained for the functions $f(z)$ given by

$$f(z) = z - \frac{(B-A)\beta(1-\alpha)}{(1+\beta B)(n+2)}z^2 \qquad z = \pm r. \tag{3.13}$$

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## ВЪРХУ НЯКОИ ПОДКЛАСОВЕ АНАЛИТИЧНИ ФУНКЦИИ С ОТРИЦАТЕЛНИ КОЕФИЦИЕНТИ


**Донка Желева Пашкулева**



Предмет на тази статия е получаването на някои резултати относно коефициенти, оценки и свойства на някои подкласове еднолистни функции в единичния кръг.