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ON HOMOCLINIC SOLUTIONS OF EXTENDED FISHER–KOLMOGOROV EQUATION

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We study the existence of homoclinic solutions of the extended Fisher–Kolmogorov equation with cubic nonlinearity and variable coefficients. An existence result is proved, using the mountain-pass theorem and the concentration-compactness principle.

1. Introduction. In this paper we study the existence of homoclinic solutions of the fourth-order equations

$$u^{iv} + pu'' + a(x)\,u - b(x)\,u^2 - c(x)\,u^3 = 0, \quad x \in \mathbf{R}, \tag{1}$$

where $u(x)$ is unknown function, p is a constant, $a(x), b(x)$ and $c(x)$ are continuous and bounded functions on $\mathbf{R}$. The equation (1), known as stationary Fisher–Kolmogorov equation, appears in several branches of Physics. The problem of finding a solution, which is homoclinic to the origin (i.e. a nontrivial function $u(x)$ such that $u(x) \to 0$ as $|x| \to \infty$) is usually related to the existence of solitary waves or to the existence of stationary solutions with finite energy, namely solutions of the Sobolev space $H^2(\mathbf{R})$.

The search for homoclinic and heteroclinic solutions is a classical problem, originating from the work of Poincaré, which has been studied from several points of view. Existence of homoclinic solutions can be obtained by analyzing the intersection properties of the stable and unstable manifolds of the fixed points. Let us recall that Devaney [5] proved that an autonomous Hamiltonian system in dimension 4, with homoclinic orbit to a saddle-focus fixed point (i.e. the linearized system at the fixed point has the eigenvalues $\pm(\alpha \pm i\omega)$, where $\alpha, \omega > 0$) is chaotic if the homoclinic orbit is the transverse intersection of the stable and unstable manifolds. The verification of the transversality for specific systems such as (1) is a difficult task.

The variational method for the existence of homoclinic and heteroclinic solutions of Fisher–Kolmogorov equation was applied by Amick & Toland [1], Buffoni [3], Peletier, Troy and Van der Vorst [6].

Since the equation (1) has a variational structure, the homoclinic solutions are critical points of the functional on $H^2(\mathbf{R})$

$$I(u) = \int_{\mathbf{R}} \left(\frac{1}{2}\left(u''^2 - pu'^2 + a(x)\,u^2\right) - \frac{1}{3}b(x)\,u^3 - \frac{1}{4}c(x)\,u^4 \right) dx. \tag{2}$$

We prove the following existence result using the mountain-pass theorem of Brezis & Nirenberg [2] and the concentration-compactness principle.

Theorem 1. *Let $a(x), b(x)$ and $c(x)$ be continuous 1-periodic functions and there exist positive constants $a_1,\ a_2,\ b,\ k_1$ and k_2 such that*

$$0 < a_1 \le a(x) \le a_2,\ |b(x)| \le b,\ 0 < k_1 \le c(x) \le k_2$$

and $p < 2\sqrt{a_1}$. Then there exists a homoclinic solution $u \in H$ of the equation (1) *which is a critical point of the functional* (2).

In the case of constant coefficients $a(x) = b(x) = 1$ and $c(x) = 0$ for $p \le -2$ Amick & Toland [1] have proved the existence of homoclinic solution of (1). Their result is extended by Buffoni [3] for $p < 2$ applying mountain-pass theorem of Brezis–Nirenberg [2] and concentration-compactness principle. This idea has been developed in Coti–Zelati, Ekeland and Séré [4] for convex Hamiltonian systems.

2. Main result. We prove the existence of homoclinic solution of (1) in the space $H^2(\mathbf{R})$ using variational method under some boundedness conditions on the coefficients $p, a(x), b(x)$ and $c(x)$. Assume that:

$$a(x), b(x) \text{ and } c(x) \text{ are continuous 1-periodic functions,} \tag{3}$$

such that there are positive constants $a_1,\ a_2,\ b,\ k_1$ and k_2 verifying

$$0 < a_1 \le a(x) \le a_2,\ |b(x)| \le b,\ 0 < k_1 \le c(x) \le k_2 \tag{4}$$

and

$$p < 2\sqrt{a_1}. \tag{5}$$

By assumption (4) it follows that the functional $I : H^2(\mathbf{R}) \to \mathbf{R}$ is Fréchet-differentiable on $H^2(\mathbf{R})$ and its Fréchet-derivative is given by

$$\langle I'(u), v\rangle = \int_{\mathbf{R}} \left(u''v'' - pu'v' + a(x)uv - b(x)u^2v - c(x)u^3v\right) dx$$

for all $v \in H^2(\mathbf{R})$. The critical point $w \ne 0, w \in H^2(\mathbf{R})$ of the functional I is a nontrivial homoclinic solution of (1). Let $\|u\|^2 = \int_{\mathbf{R}} \left(u''^2 + u'^2 + u^2\right) dx$ be the norm of the Sobolev space $H := H^2(\mathbf{R})$.

Lemma 1. *Let $a(x)$ and p satisfy assumptions* (4) *and* (5). *Then there exists a constant $c_1 > 0$ such that*

$$\int_{\mathbf{R}} \left(u''^2 - pu'^2 + a(x)u^2\right) dx \ge c_1 \|u\|^2. \tag{6}$$

Proof. If $p < 0$ it is clear that (6) is satisfied with $c_1 = \min(-p, a_1, 1)$. Let us suppose

$$0 \le p < 2\sqrt{a_1}. \tag{7}$$

Let $\hat{u}(\xi)$ be the Fourier transform of $u(x) \in H^2(\mathbf{R})$. Suppose that $k \in (0,3)$ is a constant such that

(8) $$(p+1)\,\xi^2 - a_1 + 1 \le \frac{k}{3}\left(1+\xi^2+\xi^4\right), \quad \forall \xi \in \mathbf{R}.$$

By Parseval's identity we obtain (6) with $c_1 = 1 - \frac{k}{3} > 0$ as follows

$$\begin{aligned}
\int_{\mathbf{R}} \left(u''^2 - pu'^2 + a(x)\,u^2\right) dx &\ge \int_{\mathbf{R}} \left(u''^2 - pu'^2 + a_1 u^2\right) dx \\
&= \int_{\mathbf{R}} \left(\xi^4 - p\xi^2 + a_1\right) |\hat{u}|^2(\xi)\, d\xi \\
&= \int_{\mathbf{R}} \left(\xi^4 + \xi^2 + 1 - (p+1)\,\xi^2 + a_1 - 1\right) |\hat{u}|^2(\xi)\, d\xi \\
&\ge \left(1 - \frac{k}{3}\right) \int_{\mathbf{R}} \left(\xi^4 + \xi^2 + 1\right) |\hat{u}|^2(\xi)\, d\xi \\
&= \left(1 - \frac{k}{3}\right) \|u\|^2 .
\end{aligned}$$

The inequality (8) is equivalent to

$$0 \le \xi^4 + \left(1 - \frac{3(p+1)}{k}\right)\xi^2 + \left(1 + \frac{3(a_1-1)}{k}\right), \quad \forall \xi \in \mathbf{R}.$$

The last inequality is satisfied provided that

$$\left(1 - \frac{3(p+1)}{k}\right)^2 - 4\left(1 + \frac{3(a_1-1)}{k}\right) \le 0$$

or

$$k^2 + 2k\,(p + 2a_1 - 1) - 3\,(p+1)^2 \ge 0.$$

Since $0 < k < 3$ we have

$$3 > k \ge 1 - p - 2a_1 + \sqrt{(p+2a_1-1)^2 + 3\,(p+1)^2} =: k_3.$$

The last inequality implies $p^2 < 4a_1$. Since we suppose $p \ge 0$ then we obtain (7). Conversely if (7) is satisfied, then we can choose $k \in [k_3, 3)$ such that (6) holds with $c_1 = 1 - \dfrac{k}{3}$. $\Box$

In the following, c_j denote positive constants. By the Sobolev embedding theorem

$$H^1(\mathbf{R}) \subset L^p(\mathbf{R}), \quad 2 \le p < \infty.$$

Let c_2 and c_3 be constants such that

(9) $$\int_{\mathbf{R}} |u(x)|^3\, dx \le c_2 \|u\|^3_{H^1(\mathbf{R})},$$

(10) $$\int_{\mathbf{R}} u^4(x)\, dx \le c_3 \|u\|^4_{H^1(\mathbf{R})}.$$

The functional I satisfies the geometric conditions of the mountain-pass theorem under assumptions (4) and (5) .

Lemma 2. *Let assumptions* (4) *and* (5) *hold. Then the functional* $I \in C^1(H)$ *satisfies conditions*

(1) *There exists* $\rho > 0$ *such that* $I(u) > 0$ *if* $\|u\| = \rho$.
(2) *There exists* $e \in H$ *such that* $\|e\| > \rho$ *and* $I(e) < 0$.

Proof. (1) From Lemma 1 (9) and (10) we have

$$\begin{aligned} I(u) &\geq \frac{1}{2}c_1\|u\|^2 - \frac{1}{3}b\int_{\mathbf{R}}|u|^3\,dx - \frac{1}{4}k_2\int_{\mathbf{R}}u^4dx \\ &\geq \frac{1}{2}c_1\|u\|^2 - \frac{1}{3}bc_2\|u\|^3 - \frac{1}{4}k_2c_3\|u\|^4 \\ &= \|u\|^2\left(c_4 - c_5\|u\| - c_6\|u\|^2\right) > 0 \end{aligned}$$

for sufficiently small $\|u\| = \rho$.

(2) Let us take $\bar{u} \in H, \quad \bar{u} > 0$ on $\mathbf{R}$. For $\lambda > 0$ we have

$$\begin{aligned} \lambda^{-2}I(\lambda\bar{u}) &= \frac{1}{2}\int_{\mathbf{R}}\left(\bar{u}''^2 - p\bar{u}'^2 + a(x)\bar{u}^2\right)dx - \frac{\lambda}{3}\int_{\mathbf{R}}b(x)\bar{u}^3dx - \frac{\lambda^2}{4}\int_{\mathbf{R}}c(x)\bar{u}^4dx \\ &\leq \frac{1}{2}\int_{\mathbf{R}}\left(\bar{u}''^2 - p\bar{u}'^2 + a(x)\bar{u}^2\right)dx + \frac{\lambda}{3}b\int_{\mathbf{R}}\bar{u}^3dx - \frac{\lambda^2}{4}k_1\int_{\mathbf{R}}\bar{u}^4dx \\ &\to -\infty \end{aligned}$$

as $\lambda \to +\infty$. Hence there exists $e = \lambda\bar{u}$ such that $I(e) < 0$. □

Proof of Theorem 1. By Lemma 2 and mountain-pass theorem of Brezis & Nirenberg [2], Theorem 1, there exists a sequence $(u_n)_n$ in H such that

(11) $$I(u_n) \to c > 0 \text{ and } \|I'(u_n)\|_{H^*} \to 0,$$

where

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)), \\ \Gamma &= \{\gamma \in C([0,1], H) : \gamma(0) = 0, \quad \gamma(1) = e\}. \end{aligned}$$

The functional I does not satisfy the Palais–Smale condition. For instance if $u_0(\cdot) \neq 0$ is a critical point of I then $u_0(\cdot + j), \quad j \in \mathbf{Z}$ is also a critical point of I but the sequence $(u_0(\cdot + j))_j$ has not any convergent subsequence in H.

We prove that the sequense $(u_n)_n$ is bounded in H. Indeed we have

$$\begin{aligned} \frac{1}{6}c_1\|u_n\|^2 &\leq \frac{1}{6}\int_{\mathbf{R}}\left(u_n''^2 - pu_n'^2 + a(x)u_n^2\right)dx \\ &= I(u_n) - \frac{1}{3}\langle I'(u_n), u_n\rangle - \frac{1}{12}\int_{\mathbf{R}}c(x)u_n^4dx \\ &\leq I(u_n) + \frac{1}{3}\|I'(u_n)\|_*\|u_n\|. \end{aligned}$$

Suppose that $(u_n)_n$ is unbounded sequence. By the last inequality and (11) we have

$$0 < \frac{1}{6}c_1 \leq \frac{I(u_n)}{\|u_n\|^2} + \frac{1}{3}\frac{\|I'(u_n)\|_*}{\|u_n\|} \to 0,$$

which leads to a contradiction. Hence the sequence $(u_n)_n$ is bounded in H

(12) $$\|u_n\| \leq c_7.$$

Therefore we have

$$|\langle I'(u_n), u_n\rangle| \to 0$$

and

$$\begin{aligned} 0 &< c \longleftarrow I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{6}\int_{\mathbf{R}} b(x)\, u_n^3 dx + \frac{1}{4}\int_{\mathbf{R}} c(x)\, u_n^4 dx \\ &\le \frac{1}{4}\int_{\mathbf{R}} \left(b\,|u_n|^3 + k_2 u_n^4\right) dx. \end{aligned}$$

Hence there exist constants c_8 and c_9 such that

$$0 < c_8 \le \int_{\mathbf{R}} \left(u_n^4 + |u_n|^3\right) dx \le c_9. \tag{13}$$

Denote u_n by u for simplicity. From the Sobolev inequality

$$\|u\|_{C[j,j+1]} \le 2\,\|u\|_{H^1(j,j+1)}, \quad j \in \mathbf{Z},$$

(12) and (13) we have

$$\begin{aligned} c_8 &\le \int_{\mathbf{R}} \left(u^4 + |u|^3\right) dx = \sum_j \int_j^{j+1} \left(u^4 + |u|^3\right) dx \\ &= \sum_j \left(\|u\|_{L^4(j,j+1)}^4 + \|u\|_{L^3(j,j+1)}^3\right) \\ &\le \sup_j \max\left(\|u\|_{L^4(j,j+1)}^2, \|u\|_{L^3(j,j+1)}\right) \sum_j \left(\|u\|_{L^4(j,j+1)}^2 + \|u\|_{L^3(j,j+1)}^2\right) \\ &\le 8 \sup_j \max\left(\|u\|_{L^4(j,j+1)}^2, \|u\|_{L^3(j,j+1)}\right) \sum_j \|u\|_{H^1(j,j+1)}^2 \\ &= 8 \sup_j \max\left(\|u\|_{L^4(j,j+1)}^2, \|u\|_{L^3(j,j+1)}\right) \|u\|_{H^1(\mathbf{R})}^2 \\ &\le 8c_7^2 \sup_j \max\left(\|u\|_{L^4(j,j+1)}^2, \|u\|_{L^3(j,j+1)}\right). \end{aligned}$$

Therefore

$$\sup_j \max\left(\|u\|_{L^4(j,j+1)}^2, \|u\|_{L^3(j,j+1)}\right) \ge \frac{c_8}{8c_7^2} =: c_{10}$$

and

$$\sup_j \|u\|_{L^4(j,j+1)} \ge \min\left(c_{10}, 1, \sqrt{c_{10}}\right) =: c_{11}.$$

Then

$$\inf_n \sup_{j\in\mathbf{Z}} \int_j^{j+1} u_n^4 dx = \inf_n \sup_{j\in\mathbf{Z}} \int_0^1 u_n^4 (x+j)\, dx \ge c_{11}^4 > 0. \tag{14}$$

Now we can apply the concentration-compactness argument, cf. [4]. By (14) we can choose a sequence $(j_n)_n$ such that

$$\liminf_{n\to\infty}\int_0^1 u_n^4\,(x+j_n)\,dx>0.$$

Let us define $v_n(\cdot)=u_n(\cdot+j_n)\in H$. We have

$$\|v_n\|=\|u_n\|\le c_7.$$

Going if necessary to a subsequence, we can assume that

(15) $$v_n \rightharpoonup v \quad \text{in} \quad H^2(\mathbf{R}),$$

(16) $$v_n \to v \quad \text{in} \quad L^2_{loc}(\mathbf{R}),$$

(17) $$v_n \to v \quad \text{in} \quad C_{loc}(\mathbf{R}),$$

(18) $$v_n \to v \quad \text{a.e.} \quad \text{on} \quad \mathbf{R}.$$

By (14) and (17) it follows that $v\neq 0$. Since the coefficients $a(x), b(x)$ and $c(x)$ are assumed to be 1-periodic we have

$$I(u_n)=I(v_n)$$

and for $w\in H$

$$\begin{aligned}|\langle I'(v_n),w(\cdot)\rangle| &= |\langle I'(u_n),w(\cdot-j_n)\rangle|\\ &\le \|I'(u_n)\|_*\,\|w(\cdot-j_n)\|\\ &= \|I'(u_n)\|_*\,\|w\|\to 0.\end{aligned}$$

Hence $I'(v_n)\to 0$ in H^*. By (15) and (17) it follows that $I'(v)=0$ and therefore $v\in H$ is a nontrivial homoclinic solution of the equation (1). □

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ХОМОКЛИНИЧНИ РЕШЕНИЯ НА ОБОБЩЕНИ УРАВНЕНИЯ НА ФИШЕР–КОЛМОГОРОВ

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Доказана е теорема за съществуване на хомоклинични решения на уравнения на Фишер–Колмогоров с кубични нелинейности и променливи коефициенти. Доказателството е основано на теоремата за хребета и метода на концентрираната компактност.