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SOME APPLICATIONS OF THE LINEAR ALGEBRA TO COMBINATORIAL PROBLEMS

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The purpose of this paper is to illustrate how some well-known facts from the linear algebra have beautiful and non-trivial applications in solving a variety of combinatorial problems. Only a few facts from the systems of linear and homogenious equations theory and Binnet-Cauchy formula are used. The clue to such problems is to describe the whole situation with a matrix or a system of linear equations in order to obtain the results by using linear algebraic theory. Not only these problems, but also many others can be solved with such techniques.

1. Theory. In order to solve the problems below, we need some facts from linear algebra.

1.1. Let us consider a system of n linear homogenious equations with n variables:

$$\left| \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{array} \right.$$

If the determinant is different from zero, then by Cramer's rule there is a unique solution of the system $x_1 = x_2 = \ldots = x_n = 0$. If $\det A = 0$ then the system has infinitely many solutions that can be described explicitely, when the matrix of the system has a rank $n-1$. In this case, the solutions are all n-tiples of the form $(c_1t, c_2t, \ldots, c_nt)$, where $c_1, c_2, \ldots, c_n$ are given constants, at least one of which different from zero and t is an arbitrary real number [1, 2].

1.2. If a system of linear homogenious equations contains more variables than equations, then there is a non-zero solution of this system [1, 2].

1.3. Binnet-Cauchy formula: If A is an arbitrary $n \times m$ matrix $(m > n)$, then the following equality holds

$$\det AA^T = \sum_B (\det B)^2,$$

where the summation index run through all $n \times n$ submatrices B of A [3].

2. Applications. At first, we will consider one classical problem from the graph theory.

Problem 1. (Cayley's Formula [3]) Prove that the number of all trees on n points is n^{n-2}. (A graph is called a "tree", when it is connected and has no cycles. Furthermore, if two such graphs are isomorphic, but their nodes are numerated differently, then they are different.)

Consider the complete graph K_n and let us orient its edges in an arbitrary way. Let $m = \dfrac{n(n-1)}{2}$, $v_1, v_2, \dots, v_n$ be the points of the graph and $e_1, \dots, e_m$ – its edges. We take the incidence point – edge matrix $A = (a_{ij})$, where $i = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$, such that $a_{ij} = 1$ if v_i is a head of e_j, $a_{ij} = -1$ if v_i is a tail of e_j, and $a_{ij} = 0$ otherwise. Now remove the last row. Denote the remaining $(n-1) \times m$ matrix by A_0. Consider all $(n-1) \times (n-1)$ submatrices B of A_0. Each such submatrix corresponds to a subgraph of K_n with $n-1$ edges. We claim that $\det B = \pm 1$, if this subgraph is a tree (from now on, we do not take into account the orientation of the edges of a tree), and $\det B = 0$, otherwise.

Use induction by n. Let G be a subgraph, corresponding to the matrix B. Suppose that there is a point v_i $(i < n)$, whose degree in G is 1. Expand the determinant by this row. The remaining $(n-2) \times (n-2)$ determinant B' corresponds to the graph $G - v_i$. It is clear that G is a tree in K_n, iff $G - v_i$ is a tree in $K_n - v_i$, which is also a complete graph. From this observation and from $|\det B| = |\det B'|$, we obtain the result in this case.

Suppose that in G there is no point v_i with degree 1. Notice that G is not a tree and since the number of the edges of G is $n-1$ of them, then it is easy to see that there is a point among $v_1, v_2, \dots, v_n$ with a degree 0 in G. If this point is among the first $n-1$, then B contains a row, which has only zeros and therefore $\det B = 0$. If $v_n = 0$, then each column of B has exactly one 1's and one (-1)'s and after adding all other rows to the first one, we obtain that $\det B = 0$. The result follows.

Now, by Binnet-Cauchy formula we have $\det A_0 A_0^T = \sum\limits_B (\det B)^2$. Therefore, the number of the trees is $\det A_0 A_0^T$. It is easy to calculate the determinant. Firstly, we observe that the product of the rows in B with numbers i and j is (-1) if $i \neq j$, and $n-1$ if $i = j$. Thus, the determinant is:

$$\begin{vmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & n & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & n \end{vmatrix} = n^{n-2}.$$

The last equation follows by adding all other rows to the first one and then adding the first one to all the others. Thus, the number of trees over K_n is n^{n-2}.

Problem 2. Consider a company of n $(n \geq 4)$ married couples. Sometimes, some of them gather in a group in order to discuss something. It is known that every two persons which were not wife and husband, were together in exactly one group. Moreover, no married couple was in one and the same group. Prove that $k \geq 2n$.

For clarity, consider n sets $A_1 = \{a_1, b_1\}, A_2 = \{a_2, b_2\}, \dots, A_n = \{a_n, b_n\}$. Furthermore, let $C_1, C_2, \dots, C_k$ be subsets of the set $\bigcup\limits_{i=1}^{n} A_i$, for which each pair (a_i, b_j), (a_i, a_j) or (b_i, b_j) belongs to exactly one of them, if $i \neq j$, and to none, if $i = j$. We have to prove that $k \geq 2n$.

Assume that $k < 2n$. We assign every variable x_i to a_i, y_i – to each b_i and t_i – to each C_i. Consider the following linear homogenious system of k equations and $2n$ variables – $x_1,\ y_1, \ldots,\ x_n,\ y_n$.

$$\left| \begin{array}{l} t_1 = 0 \\ t_2 = 0 \\ \vdots \\ t_n = 0 \end{array} \right.$$

By 1.2) and the assumption $k < 2n$ it follows that this system has a non-zero solution. Let $d_1,\ d_2, \ldots,\ d_n$ and $e_1,\ e_2, \ldots,\ e_n$ be the number of sets which contain $a_1,\ a_2, \ldots,\ a_n$ and $b_1,\ b_2, \ldots,\ b_n$ respectively. It is clear that none of these numbers is 1, because of the given conditions. If one of these numbers is 2 (say $d_1 = 2$), then we observe that a_1 is contained in exactly two sets (C_1 and C_2) with $n-1$ elements each. But each element from C_1 must be in a pair with $n-2$ elements from C_2 in some subset. Moreover, no two different pairs can be in one and the same subset, since each non-married couple belongs to exactly one subset. Hence

$$k \geq (n-1)(n-2) + 2 \geq 2n.$$

It suffices to consider the non-trivial case, when $d_i \geq 3$ and $e_i \geq 3$ for all i. In this case, using the given condition we may write

$$\begin{aligned} 0 &= t_1^2 + \ldots + t_n^2 = (d_1x_1^2 + \ldots + d_nx_n^2) + (e_1y_1^2 + \ldots + e_ny_n^2) + \\ &+ \sum_{i \neq j} 2(x_iy_j + x_ix_j + y_iy_j) = (d_1-1)x_1^2 + \ldots + (d_n-1)x_n^2 + \\ &+ (e_1-1)y_1^2 + \ldots + (e_n-1)y_n^2 + \left(\sum_{i=1}^{n}(x_i+y_i)\right)^2 - \sum_{i=1}^{n} x_iy_i = \\ &= (d_1-2)x_1^2 + \ldots + (d_n-2)x_n^2 + (e_1-2)y_1^2 + \ldots + (e_n-2)y_n^2 + \\ &+ \left(\sum_{i=1}^{n}(x_i+y_i)\right)^2 + \sum_{i=1}^{n}(x_i-y_i)^2 > 0, \end{aligned}$$

which is obviously a contradiction. Hence $k \geq 2n$.

The clue of this problem is how to use in the best possible way the given condition that each non-married couple belongs to exactly one subset. As one can see, the sum of the squares of the numbers $t_1, \ldots,\ t_n$ includes the whole information.

Problem 3. Each side of a regular 2^k-gon is colored either in white or in black. In each step, a side, whose neighboring sides are in different colors, is colored white, and each side with neighbors of one and the same color becomes black. Prove that after 2^{k-1} steps all the sides will be white, and that 2^{k-1} is the best possible.

Let us assign 0 (mod 2) to each white side and 1 (mod 2) to each black one. Moreover, we number all the sides of the polygon with the integers from 1 to 2^k. Thus, in each moment, the current position may be described with a vector $v_i = (a_1,\ a_2, \ldots,\ a_{2^k})$. It is clear that on the next step the vector v_{i+1} will be $(a_{2^k}+a_2,\ a_1+a_3, \ldots,\ a_{2^k-1}+a_1)$ where all the numbers are taken (mod 2). We can write $v_{i+1} \sim (a_{2^k}+a_2,\ a_1+a_3, \ldots,\ a_{2^k-1}+a_1)$.

Consider the following $2^k \times 2^k$ matrix:

$$A = \begin{pmatrix} 0\,1\,0\,\ldots\,1 \\ 1\,0\,1\,\ldots\,0 \\ \vdots \quad \vdots \quad \vdots \\ 1\,0\,0\,\ldots\,0 \end{pmatrix}.$$

Then $v_{i+1} \sim v_i A$. But $A = M + M^{-1}$, where

$$M = \begin{pmatrix} 0\,1\,0\,\ldots\,0 \\ 0\,0\,1\,\ldots\,0 \\ \vdots \quad \vdots \quad \vdots \\ 0\,0\,0\,\ldots\,1 \\ 1\,0\,0\,\ldots\,0 \end{pmatrix}.$$

It is easy to see that:

$$v_{2^{k-1}} \sim v_0(M + M^{-1})^{2^{k-1}} = v_0 \left(\sum_{i=0}^{2^{k-1}} \binom{2^{k-1}}{i} M^{2i-2^{k-1}} \right).$$

According to the Lucas theorem, $\binom{2^{k-1}}{i}$ is divisible by 2 when $i = 1, \ldots, 2^{k-1} - 1$. Furtermore, since $M^{2^k} = E$, then $M^{2^{k-1}} = M^{-2^{k-1}}$. In this way we obtain $v_{2^{k-1}} \sim (0, 0, \ldots, 0)$ which means that after the 2^{k-1}-th step all the sides will be white. In order to show that 2^{k-1} is the least possible integer, it suffices to consider (1,0,0,…,0) as an original vector.

Problem 4. $2n + 1$ real numbers are given, such that if among of these numbers is removed, the others can be divided into 2 groups with n numbers in each and with equal sums. Prove that all the numbers are equal.

Let us denote the given numbers by $a_1, \ldots,\ a_{2n+1}$. We can write the conditions of the problem as a system of linear homogenious equations with $2n + 1$ variables:

$$\left| \begin{array}{l} 0a_1 + \epsilon_{12}a_2 + \ldots + \epsilon_{1,2n+1}a_{2n+1} = 0 \\ \epsilon_{21}a_1 + 0a_2 + \ldots + \epsilon_{2,2n+1}a_{2n+1} = 0 \\ \vdots \\ \epsilon_{2n+1,1}a_1 + \epsilon_{2n+1,2}a_2 + \ldots + 0a_{2n+1} = 0 \end{array} \right.$$

where $\epsilon_{ij} = \pm 1$ and in each row there is exactly n 1's and n (-1)'s.

Notice that the determinant of this system is 0, which can be proved by adding all other columns to the first. We claim that the rank of the matrix of the system is $2n$. Consider the minor formed by the first $2n$ rows and the first $2n$ columns of the matrix. Its determinant is

$$\begin{vmatrix} 0\ \epsilon_{12}\ \ldots\ \epsilon_{1,2n} \\ \epsilon_{21}\ 0\ \ldots\ \epsilon_{2,2n} \\ \vdots \\ \epsilon_{2n,1}\ \epsilon_{2n,2}\ \ldots\ 0 \end{vmatrix}$$

This determinant is an odd number. We will show this by making the following observation: The determinant is equivalent modulo 2 with the number of permutations

P_0 of $1, 2, \ldots, 2n$ without any fixed point. This number can be calculated easily by the inclusion-exclusion principle: if X is a subset of $\{1, 2, \ldots, 2n\}$ then the number P_X of those permutations for which the numbers from X are fixed points is $(2n - |X|)!$. By the inclusion-exclusion principle:

$$\begin{aligned} P_0 &= \sum_{X\subseteq\{1,2,\ldots,2n\}} (-1)^{|X|} P_X = \sum_{X\subseteq\{1,2,\ldots,2n\}} (-1)^{|X|}(2n - |X|)! = \\ &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (2n-k)! = \sum_{k=0}^{2n} (-1)^k \frac{2n!}{k!}, \end{aligned}$$

which is evidently an odd number.

Hence, the determinant of this minor is different from zero and it follows that the rank of the matrix is $2n$. From 1.1. the solutions are all $2n$-tiples $(c_1 t, c_2 t, \ldots, c_{2n+1} t)$ where t is an arbitrary real number and $c_1, c_2, \ldots, c_{2n+1}$ are given constants. Since each row of the matrix has n 1's and n (-1)'s, then $(t, t, \ldots, t)$ is a solution. It follows that $c_1 = c_2 = \ldots = c_{2n+1}$, which solves the problem.

A similar idea appears in the following problem:

Problem 5. Let p be a prime number and $p+1$ real numbers be given with the following property: if we remove an arbitrary number, the others can be divided into several groups (at least two), such that the average mean of the numbers of each group is one and the same. Prove that the numbers are equal.

Let $a_1,\ a_2, \ldots,\ a_{p+1}$ be the given numbers. It is clear that if c is an arbitrary real number, then $a_1 - c,\ a_2 - c, \ldots,\ a_{p+1} - c$ have the same property as $a_1,\ a_2, \ldots,\ a_{p+1}$. Therefore, we consider that the sum of the given numbers is 0. Let us remove, for instance, the number a_{p+1}. Then the other numbers can be divided into several groups, such that the average means of the groups are equal. If m is the average mean of the numbers of each group, then it is easy to see that $\dfrac{a_{p+1}}{p} = -m$. Consider one of these groups. If it contains k numbers: $a_{s_1},\ a_{s_2}, \ldots,\ a_{s_k}$ then, by considering $a_1,\ a_2, \ldots,\ a_{p+1}$ as variables we obtain the following linear homogenious equation:

$$\frac{1}{k} a_{s_1} + \frac{1}{k} a_{s_2} + \ldots + \frac{1}{k} a_{s_k} - \frac{1}{p} a_{p+1} = 0.$$

Write a system of $p+1$ such equations (removing $a_1, \ldots, a_{p+1}$ respectively). The determinant is:

$$D = \begin{vmatrix} \frac{1}{p} & c_{12} & \cdots & c_{1,p+1} \\ c_{21} & \frac{1}{p} & \cdots & c_{2,p+1} \\ \vdots & & & \\ c_{p+1,1} & c_{p+1,2} & \cdots & \frac{1}{p} \end{vmatrix},$$

where $c_{ij} = 0$ or $c_{ij} = \dfrac{1}{m}$, where $m < p$. It is not difficult to see that $D = \dfrac{1}{p^{p+1}} + \dfrac{M}{Np^l}$, where $l < p$. It follows that the determinant is not zero and hence, the system has only the zero solution. Hence, the numbers $a_1,\ a_2, \ldots,\ a_{p+1}$ are equal.

4. Conclusion. The above examples illustrate the variety of applications of some facts from linear algebra in combinatorial problems. These problems have other solutions with the usage of some combinatorial observations or facts from the number theory.

However, linear algebraic methods give clarity in the description of the situations and may serve as an effective tool.

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НЯКОИ ПРИЛОЖЕНИЯ НА ЛИНЕЙНАТА АЛГЕБРА В КОМБИНАТОРИКАТА

Димитър Петков Жечев

Целта на настоящия доклад е да илюстрира как някои фундаментелни твърдения от линейната алгебра намират красиви и нетривиални приложения в множество комбинаторни проблеми. За решаване на поставените примери се използват някои факти от теорията на системите линейни хомогенни уравнения и формулата на Бине-Коши. Основна насока в такива задачи е да се опише ситуацията с матрица или система линейни уравнения с цел да се използват известни резултати. Не само разглежданите задачи, но и много други могат да се решат с предложения подход.