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# LIMIT THEOREMS FOR RENEWAL, REGENERATION AND BRANCHING PROCESSES * 

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The paper presents several new limit theorems for a class of non-negative alternating regenerative processes and for the embedded renewal processes with two alternative states. These results give new method for further investigations of Bellman-Harris branching processes with state-dependent immigration. The limiting distributions for these processes are obtained in the critical case with infinite offspring variance.

1. Introduction. It is well known that the renewal and regenerative processes play an important role in many asymptotic investigations and real applications, see, for example $[14,15,6,16,8,3,2,4,13,7]$.

The aim of the paper is to present several limit theorems for a class of non-negative alternating regenerative processes, proved in [12].

The dynamics of these processes on a period of regeneration can be described as follows: the process stays at zero random time (waiting period) and after that it develops following some stochastic laws until it hits zero (living period or work time). The investigation of the regenerative processes with two states needs the investigation of the corresponding two states renewal processes.

So, at first we investigate the asymptotic behaviour of the spent time associated with an alternating renewal process. The asymptotic results generalize those obtained by Dynkin [5], Lamperti [9] and Erickson [6] (see also [16, 4] and [1]). On the other hand, in alternating case new phenomena are obtained not observed in the classical renewal processes (see Renewal Theorems 1 and 2).

We apply the results for alternating regenerative processes to prove several new limit theorems for Bellman-Harris branching processes with state-dependent immigration defined in $[10,11]$.
2. Definitions. Let on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ be given:
i) A set $X=\left\{X_{i}, i=1,2, \ldots\right\}$ of independent, identically distributed (i.i.d.), positive random variables (r.v.) with distribution function (d.f.) $A(t)=\mathbf{P}\left\{X_{i} \leq t\right\}$.
ii) An independent of $X$ set $Z=\left\{\left(T_{i},\left(Z_{i}(t)>0, t \in\left[0, T_{i}\right), Z_{i}\left(T_{i}\right)=0\right)\right), i=1,2, \ldots\right\}$ of i.i.d. pairs of a nonnegative r.v. $T_{i}$ and a measurable stochastic process $Z_{i}(t)$ with state space $\left(\mathbf{R}^{+}, \mathcal{B}^{+}\right)$, where $\mathbf{R}^{+}=[0, \infty)$, and $\mathcal{B}^{+}$is the Borel $\sigma$-field. The r.v. $T_{i}$ with d.f. $B(t)=\mathbf{P}\left\{T_{i} \leq t\right\}$ is called the life-period of the process $Z_{i}(t)$.

[^0]Denote by $Y_{i}=X_{i}+T_{i}, \quad i=1,2, \ldots$ the sequence of i.i.d. non-negative r.v.'s with d.f. $C(t)=(A * B)(t) \equiv \int_{0}^{t} A(t-u) B(d u)$.

We will denote by the same letters the d.f. and the corresponding measures on $\mathbf{R}$ and will assume the d.f. right continuous. For the integrals we assume $\int_{a}^{b} \equiv \int_{a-}^{b+}$.

Define the ordinary renewal process $N(t)=\max \left\{n: S_{n} \leq t<S_{n+1}\right\}$, where $S_{0}=0$, $S_{n+1}=S_{n}+Y_{n+1}, n=0,1,2, \ldots$. The sequence $\left\{S_{n}, S_{n+1}^{\prime}\right\}_{n=0}^{\infty}$ where $S_{n+1}^{\prime}=S_{n}+$ $X_{n+1}, n=0,1,2, \ldots$ is called an alternating renewal process. In the terms of reliability theory one can interpret $X_{i}$ as the time for the installation (or repairing time) of $i-$ th element in some system and $T_{i}$ as the work time of the same element. So, in the process $\left\{S_{n}, S_{n+1}^{\prime}\right\}$ there are two types of renewal events: $S_{n}$ the beginning of the installation and $S_{n+1}^{\prime}=S_{n}+X_{n+1}$ the beginning of the work of the $(n+1)-$ th element.

For the ordinary renewal process the behaviour of the spent life time $\tau(t)=t-S_{N(t)}$ is well studied (see e.g. [16],[6]). We define now the spent time for the alternating renewal process $\left\{S_{n}, S_{n+1}^{\prime}\right\}_{n=0}^{\infty}$,

$$
\sigma(t)=t-S_{N(t)+1}^{\prime}=t-S_{N(t)}-X_{N(t)+1}, \quad t \geq 0
$$

Note that we obtain the classical renewal process if all $X_{i}$ are identically equal to zero. In this case $\sigma(t)$ is always non-negative. It is obviously that for the alternating renewal process, $\sigma(t)$ can take both negative and positive values.

We will use the process $\sigma(t)$ (spent time) to construct an alternating regenerative process $Z(t), t \geq 0$, as follows

$$
Z(t)= \begin{cases}Z_{N(t)+1}(\sigma(t)), & \sigma(t) \geq 0  \tag{2.1}\\ 0, & \sigma(t)<0\end{cases}
$$

The process $Z(t)$ is a classical regenerative process, if we consider only the renewal epoch $\left\{S_{n}\right\}_{n=0}^{\infty}$, but it can be interpret as an alternating regenerative process, if we consider also the epoch $\left\{S_{n+1}^{\prime}=S_{n}+X_{n+1}\right\}_{n=0}^{\infty}$. So, $Z(t)$ can be described as follows: $Z(t)=0$ for $t \in\left[0, X_{1}\right), Z(t)=Z_{1}\left(t-X_{1}\right)>0$ for $t \in\left[X_{1}, X_{1}+T_{1}\right), \ldots, Z(t)=0$ for $t \in\left[S_{n}, S_{n}+X_{n+1}\right), Z(t)=Z_{n+1}\left(t-S_{n}-X_{n+1}\right)>0$ for $t \in\left[S_{n}+X_{n+1}, S_{n+1}\right)$, and so on.
3. Basic assumptions. We will suppose that $A(0)=0, B(0)=0$, and $A(t)$ and $B($.$) are non-lattice d.f. and some of the following basic conditions are satisfied:$
(A.1) $m_{A} \equiv \mathbf{E} X_{i}<\infty$.
(A.2) $\mathbf{E} X_{i}=\infty, \bar{A}(t)=1-A(t) \sim t^{-\alpha} L_{A}(t), t \rightarrow \infty, \quad \alpha \in\left(\frac{1}{2}, 1\right]$, and for each $h>0$ fixed $A(t)-A(t-h)=O(1 / t), \quad t>0$.
(B.1) $m_{B} \equiv \mathbf{E} T_{i}<\infty$.
(B.2) $\mathbf{E} T_{i}=\infty, \quad \bar{B}(t)=1-B(t) \sim t^{-\beta} L_{B}(t), \quad t \rightarrow \infty, \quad \beta \in\left(\frac{1}{2}, 1\right]$.

The functions $L_{A}($.$) and L_{B}($.$) are slowly varying at infinity.$
(C.0) For $x \geq 0$ there exists the limit

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{Z_{1}(t)}{M(t)} \leq x \right\rvert\, T_{1}>t\right\}=D(x)
$$

where $M(t)$ is a positive, non-decreasing function, regularly varying at infinity with exponent $\gamma \geq 0$, and $D(x)$ is a proper d.f. on $\mathbf{R}^{+}$.

The following notations $m_{A}(t)=\int_{0}^{t} \bar{A}(u) d u$ and $m_{B}(t)=\int_{0}^{t} \bar{B}(u) d u$ will be used.
4. Limit theorems for the spent work time $\sigma(t)$. If the both means $\mathbf{E} X_{i}$ and $\mathbf{E} T_{i}$ are finite the limit behaviour of $\sigma(t)$ can be easily obtained from the classical results
(see e.g. [16], Sect.11.9). For this reason we assume that at least one of these means is infinite. Moreover, we will always assume that there exist the limit

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\bar{A}(t)}{\bar{B}(t)}=c, \quad 0 \leq c \leq \infty . \tag{4.1}
\end{equation*}
$$

The next two theorems contain the basic limit results for $\sigma(t)$.

Renewal Theorem 1. Assume (B.2) and (4.1) with $0 \leq c<\infty$. If (A.1) or (A.2) is fulfilled, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{\sigma(t) \geq 0\}=\frac{1}{1+c}
$$

(i) If additionally $1 / 2<\beta<1$ in (B.2), then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{\sigma(t)}{t} \leq x \right\rvert\, \sigma(t) \geq 0\right\}=\frac{\sin \pi \beta}{\pi} \int_{0}^{x} u^{-\beta}(1-u)^{1-\beta} d u
$$

(ii) If additionally $\beta=1$ in (B.2), then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{m_{B}(\sigma(t))}{m_{B}(t)} \leq x \right\rvert\, \sigma(t) \geq 0\right\}=x
$$

Renewal Theorem 2. Assume (A.2) and (4.1) with $c=\infty$. If additionally (B.1) or (B.2) is fulfilled, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{\sigma(t) \geq 0\}=0
$$

(i) If only (B.1) holds, then for $x>0$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{\sigma(t) \leq x \mid \sigma(t) \geq 0\}=\frac{m_{B}(x)}{m_{B}}
$$

(ii) If only (B.2) with $1 / 2<\beta<1$ holds, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{\sigma(t)}{t} \leq x \right\rvert\, \sigma(t) \geq 0\right\}=\frac{\int_{0}^{x} u^{-\beta}(1-u)^{\alpha-1} d u}{\mathbf{B}(\alpha, 1-\beta)}
$$

where $\mathbf{B}(.,$.$) is the standard beta function.$
(iii) If only (B.2) with $\beta=1$ holds, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{m_{B}(\sigma(t))}{m_{B}(t)} \leq x \right\rvert\, \sigma(t) \geq 0\right\}=x
$$

The proofs of these theorems are based on the equations of renewal type for the distribution of $\sigma(t)$, and some results correspond to the key-renewal theorem in the infinite mean case.
5. Basic Regeneration Theorem. In this section we formulate a theorem which describes the limiting behaviour of the process $Z(t)$.

Basic Regeneration Theorem (BRT). Assume conditions (2.1), (4.1) and (C.0).

1) Let (B.2) holds with $1 / 2<\beta<1$ :

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(i) If additionally (A.1) or (A.2) is fulfilled, and $0 \leq c<\infty$, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\frac{Z(t)}{M(t)} \leq x\right\}=\frac{c}{1+c}+\frac{G_{1}(x)}{1+c}
$$

where $G_{1}(x)=\frac{\sin \pi \beta}{\pi} \int_{0}^{1} D\left(x u^{-\gamma}\right)(1-u)^{1-\beta} u^{-\beta} d u$ is a d.f. on $\mathbf{R}^{+}$.
(ii) If (A.2) is fulfilled and $c=\infty$, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{Z(t)}{M(t)} \leq x \right\rvert\, Z(t)>0\right\}=G_{2}(x)
$$

where $G_{2}(x)=\frac{1}{\mathbf{B}(1-\beta, \alpha)} \int_{0}^{1} D\left(x u^{-\gamma}\right)(1-u)^{\alpha-1} u^{-\beta} d u$ is a d.f. on $\mathbf{R}^{+}$.
2) Let (B.2) holds with $\beta=1, D(0)=0$ and $\gamma>0$ :
(iii) If additionally (A.1) or (A.2) is fulfilled, and $0 \leq c<\infty$, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\frac{m_{B}\left(M^{-1}(Z(t))\right)}{m_{B}(t)} \leq x\right\}=\frac{c}{1+c}+\frac{x}{1+c}
$$

where $M^{-1}($.$) is the inverse function of M($.$) .$
(iv) If (A.2) is fulfilled and $c=\infty$, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{m_{B}\left(M^{-1}(Z(t))\right)}{m_{B}(t)} \leq x \right\rvert\, Z(t)>0\right\}=x
$$

3) Let (B.1) and (A.2) hold. Then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

Remark 1. If one assume (A.1) and (B.1) only, then the process $Z(t)$, with renewal epochs $\left\{S_{n}\right\}$, is positive recurrent and without any other assumptions by the classical regenerative theorem (See e.g. [13], Theorem 2.1) there exists

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x\}=\frac{1}{m_{A}+m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z(u) \leq x, X_{1}+T_{1}>u\right\} d u
$$

The proof of the BRT is based on the properties of the limiting distributions of $\sigma(t)$, given in Renewal Theorems 1 and 2, and certain results for the convergence in distribution.
6. Branching processes with state - dependent immigration. In this section we assume that on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$ instead of the set $Z$ are given:
i) The set $\tilde{Z}=\left\{\left(Z_{i j}(t), t \geq 0\right), \quad i=1,2, \ldots, j=1,2, \ldots.\right\}$ of independent, identically distributed (i.i.d) Bellman-Harris branching processes with p.g.f. of the offspring of one particle $f(s)$, and the d.f. of the life-length of one particle $G(t)$. We denote by $F(t, s)$ the p.g.f. of $Z_{i j}(t)$.
ii) The set $W=\left\{W_{i}, i=1,2, \ldots\right\}$ of i.i.d., positive, integer valued random variables (r.v.) with p.g.f. $g(s)=\mathbf{E}\left\{s^{W_{i}}\right\}, s \in[0,1]$.

Define the sequence $\left\{Z_{i}(t), t \geq 0, i=1,2, \ldots\right\}$ of i.i.d. branching processes starting
with positive random number of ancestors $W_{i}$ at time $t=0$ as follows

$$
Z_{i}(t)=\sum_{j=1}^{W_{i}} Z_{i j}(t), \quad t \geq 0
$$

and denote by $T_{i}$ the life period of the process $Z_{i}(t)$, i.e. $T_{i}$ is a r.v. such that $Z_{i}(0)=$ $W_{i}>0, Z_{i}(t)>0$ for $t \in\left[0, T_{i}\right), Z_{i}\left(T_{i}\right)=0$.

It is clear that the r.v. $T_{i}, i=1,2, \ldots$ are independent and identically distributed.
Now, if we use the pairs $\left(Z_{i}(t), T_{i}\right)$ in the definition of the process $Z(t)$ then $Z(t)$ will be a Bellman-Harris branching process with state-dependent immigration. (Cf. [10,11].)

In what follows we will assume for the set $X$ the conditions (A.1) and (A.2), but instead of (B.1), (B.2) and (C.0) we will assume the following conditions:

1) For the offspring p.g.f.

$$
\begin{equation*}
h(s)=s+(1-s)^{1+\delta} L\left(\frac{1}{1-s}\right), \quad s \uparrow 1, \delta \in(0,1], \tag{6.1}
\end{equation*}
$$

where $L(t)$ is a s.v.f. at infinity.
2) For the particle life d.f.

$$
\begin{equation*}
G(0)=0, \quad G(t) \text { is non-lattice and } r=\int_{0}^{\infty} x d G(x)<\infty \tag{6.2}
\end{equation*}
$$

and there exists

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{n(1-G(n))}{1-h_{n}(0)}=d, \quad 0 \leq d \leq \infty \tag{6.3}
\end{equation*}
$$

where as usual $h_{n}(s)$ is the $n$-fold iteration of $h(s): h_{0}(s)=s, h_{n+1}(s)=h\left(h_{n}(s)\right)$.
In some cases the following condition will also be assumed:

$$
\begin{equation*}
1-G(t) \sim t^{-\kappa} L_{G}(t), \quad t \rightarrow \infty, \quad \kappa>0 . \tag{6.4}
\end{equation*}
$$

3) For the immigration in the state zero it is assumed one of the following conditions:

$$
\begin{equation*}
m_{W}=\mathbf{E} W_{i}=g^{\prime}(1)<\infty, \tag{6.5}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{E} W_{i}=\infty, \quad g(s) \sim 1-(1-s)^{\theta} L_{B}\left(\frac{1}{1-s}\right), \quad \theta \in(0,1] \tag{6.6}
\end{equation*}
$$

where $L_{B}(t)$ is a s.v.f. at infinity.
Remark 2. The condition (6.1) determines $Z_{i j}(t)$ as critical Bellman - Harris processes with possibly infinite variance. Since the case of finite variance is considered in $[10,11]$, then we will suppose now that $h^{\prime \prime}(1-)=\infty$.

In the theorems which are stated below we will use the following Laplace transforms:

$$
\hat{D}_{\rho, \delta}(\lambda)=1-\lambda\left(1+\frac{\rho^{\delta}(0) \lambda^{\delta}}{r}\right)^{-1 / \delta}, \hat{D}_{\delta}(\lambda)=1-\lambda\left(1+\lambda^{\delta}\right)^{-1 / \delta}, \quad \lambda>0
$$

and the following p.g.f.:

$$
f_{\rho}(s)=1-\frac{\rho(s)}{\rho(0)}, \quad f_{\delta}(s)=1-(1-s)^{1 /(1+\delta)}, \quad s \in[0,1] .
$$

Remark 3. The function $\rho(s)$ is the unique positive solution of the equation (see [20])

$$
\rho(s)^{1+\delta}-r \rho(s)-(1-s) d \delta=0, \quad s \in[0,1] .
$$

where $\delta \in(0,1]$ is defined by (6.1), $0<r<\infty$ by (6.2) and $0<d<\infty$ by (6.3). The Laplace transforms and the p.g.f. given above arise for the corresponding limit distributions of the process $Z_{i j}(t)$, obtained by Vatutin in $[17,18,19,20]$ under the conditions (6.1)-(6.4).

Further on, we will always suppose that there exists:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\bar{A}(t)}{\bar{B}(t)}=c, \quad 0 \leq c \leq \infty \tag{6.7}
\end{equation*}
$$

We will use the basic notation $Q(t)=1-F(t, 0)=\mathbf{P}\left\{Z_{i j}(t)>0\right\}$. Note that in the case $0 \leq d<\infty$ one has $Q(t) \sim t^{-1 / \delta} L_{Q}(t)$, where $L_{Q}(t)$ is a s.v.f. as $t \rightarrow \infty$ (see [20]).

Now, we are able to present some limit theorems for the Bellman-Harris branching process $Z(t)$ with state-dependent immigration. The proofs of these theorems consist of the examination of the conditions of BRT.

Theorem 6.1. Assume (6.1), (6.2), (6.3) with $d=0,(6.5)$ and (6.7).

1) Let $\delta=1$ and $\mathbf{E} T_{i}=\infty$.
(i) If additionally $0 \leq c<\infty$, and (A.1) or (A.2) is fulfilled, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\frac{m_{B}\left(Q_{1}^{-1}(Z(t))\right)}{m_{B}(t)} \leq x\right\}=\frac{c+x}{1+c}, \quad x \in(0,1)
$$

where $Q_{1}^{-1}(t)$ is the inverse function of $1 / Q(t)$.
(ii) If additionally $c=\infty$, and (A.2) holds, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{m_{B}\left(Q_{1}^{-1}(Z(t))\right)}{m_{B}(t)} \leq x \right\rvert\, Z(t)>0\right\}=x, \quad x \in(0,1)
$$

2) Let $\delta \leq 1$ and $m_{B}=\mathbf{E} T_{1}<\infty$.
(i) If additionally (A.2) holds, then for $x \geq 0$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

(ii) If additionally (A.1) is satisfied, then for $x \geq 0$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x\}=\frac{m_{A}}{m_{A}+m_{B}}+\frac{1}{m_{A}+m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

Theorem 6.2. Assume (6.1), (6.2), (6.3) with $d=0$, (6.6) and (6.7). Let $\beta=\theta / \delta$.

1) If $1 / 2<\beta<1$ and, either (A.2) with $\alpha>\beta$ or (A.1) is fulfilled, then for $x \geq 0$,
(6.8) $\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x\}=\frac{\sin \pi \beta}{\pi} \int_{0}^{\infty} D_{\delta, \theta}\left(x u^{-1 / \delta}\right)(1-u)^{1-\beta} u^{-\beta} d u=G_{1}(x)$,
where the d. $f . D_{\delta, \theta}(x)$ has a Laplace transform $\hat{D}_{\delta, \theta}(\lambda)=1-\left(1-\hat{D}_{\delta}(\lambda)\right)^{\theta}, \quad \lambda>0$.
2) If (A.2) holds and $1 / 2<\alpha<\beta<1$, then for $x \geq 0$,

$$
\begin{gather*}
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x \mid Z(t)>0\}  \tag{6.9}\\
=\frac{1}{\mathbf{B}(1-\beta, \alpha)} \int_{0}^{\infty} D_{\delta, \theta}\left(x u^{-1 / \delta}\right)(1-u)^{\alpha-1} u^{-\beta} d u
\end{gather*}
$$

3) Let $\beta=1$ and $\mathbf{E} T_{i}=\infty$ :
(i) If additionally (A.1) or (A.2) is fulfilled and $0 \leq c<\infty$, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\frac{m_{B}\left(Q_{1}^{-1}(Z(t))\right)}{m_{B}(t)} \leq x\right\}=\frac{c+x}{1+c}, \quad x \in(0,1)
$$

(ii) If additionally (A.2) holds and $c=\infty$, then for $0<x<1$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{\left.\frac{m_{B}\left(Q_{1}^{-1}(Z(t))\right)}{m_{B}(t)} \leq x \right\rvert\, Z(t)>0\right\}=x
$$

where $Q_{1}^{-1}(t)$ is the inverse function of $1 / Q(t)$.
4) If $\beta \geq 1, m_{B}=\mathbf{E} T_{1}<\infty$ and (A.2) is fulfilled, then for $x \geq 0$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

5) Let (A.2) holds and $1 / 2<\alpha=\beta<1$ :
(i) If $0 \leq c<\infty$, then for $x \geq 0$,

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x\}=\frac{c+G_{1}(x)}{1+c}
$$

where $G_{1}(x)$ is from (6.8).
(ii) If $c=\infty$ then the limit (6.9) is fulfilled.

Theorem 6.3. Assume (6.1), (6.2), (6.3) with $0<d<\infty$ and (6.7).
A) Suppose additionally (6.5), (A.2) and, $0<\delta \leq 1$ but $\mathbf{E} T_{1}<\infty$. Then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u \tag{6.10}
\end{equation*}
$$

B) Suppose additionally (6.6) and denote $\beta=\theta / \delta$ :

1) If $1 / 2<\beta<1$ and, either (A.1) or (A.2) with $\beta<\alpha \leq 1$ hold, then
(6.11) $\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x\}=\frac{\sin \pi \beta}{\pi} \int_{0}^{1} D_{\rho, \delta, \theta}\left(x u^{-1 / \delta}\right)(1-u)^{1-\beta} u^{-\beta} d u=G_{1}(x)$,
where the d.f. $D_{\rho, \delta, \theta}(x)$ has a Laplace transform $\hat{D}_{\rho, \delta, \theta}(\lambda)=1-\left(1-\hat{D}_{\rho, \delta}(\lambda)\right)^{\theta}, \quad \lambda>0$.
2) If (A.2) with $1 / 2<\alpha<\beta<1$ is fulfilled, then

$$
\begin{gather*}
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x \mid Z(t)>0\}  \tag{6.12}\\
=\frac{1}{\mathbf{B}(1-\beta, \alpha)} \int_{0}^{1} D_{\rho, \delta, \theta}\left(x u^{-1 / \delta}\right)(1-u)^{\alpha-1} u^{-\beta} d u
\end{gather*}
$$

3) Let (A.2) with $1 / 2<\alpha=\beta<1$ be fulfilled:
(i) If $0 \leq c<\infty$ then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t) Q(t) \leq x\}=\frac{c+G_{1}(x)}{1+c}
$$

where $G_{1}(x)$ is given in (6.11).
(ii) If $c=\infty$ then (6.12) holds.
4) Let (A.2) holds. If $\beta \geq 1$ and $\mathbf{E} T_{1}<\infty$, then (6.10) is fulfilled.

Comment 1. It is not difficult to see that $\hat{D}_{\rho, \delta}(\infty)=1-\frac{r^{1 / \delta}}{\rho(0)}=D_{\rho, \delta}(0)>0$ and $\hat{D}_{\rho, \delta, \theta}(\infty)=1-\frac{r^{\theta / \delta}}{\rho^{\theta}(0)}=D_{\rho, \delta, \theta}(0)>0$.

This explains why it is not possible to apply the BRT, 2). Thus the case $\beta=1$ and $\mathbf{E} T_{1}=\infty$ is an open problem.

Comment 2. Under the conditions of the Theorem Vatutin [19,20] proves that

$$
\lim _{t \rightarrow \infty} \mathbf{P}\left\{Z_{i j}(t)=\mid Z_{i j}(t)>0\right\}=p_{n}=\mathbf{P}\{\xi=n\}
$$

where the p.g.f. $\mathbf{E} s{ }^{\xi}=f_{\rho}(s)$. It is not difficult to see that $f_{\rho}(1)=1-\rho(1) / \rho(0)=$ $1-r^{1 / \delta} / \rho(0)=\mathbf{P}\{\xi<\infty\}<1$, i.e. the limit distribution is improper one. Therefore, it is not possible to apply the BRT because the condition (C.0) is not fulfilled. So, it is an open problem to obtain the corresponding limit theorem for the process $Z(t)$.

Theorem 6.4. Assume (6.1), (6.2), (6.3) with $d=\infty$, (6.5), (6.7) and (6.4). Denote $\beta=\kappa /(1+\delta):$

1) If $1 / 2<\beta<1$ and, either (A.1) or (A.2) with $\beta<\alpha \leq 1$ is fulfilled, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n\}=\mathbf{P}\left\{Z_{\infty}=n\right\}
$$

where the p.g.f. $\mathbf{E} s^{Z_{\infty}}=f_{\delta}(s)$.
2) If (A.2) with $1 / 2<\alpha<\beta<1$ is fulfilled, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n \mid Z(t)>0\}=\mathbf{P}\left\{Z_{\infty}=n\right\} \tag{6.13}
\end{equation*}
$$

3) Let (A.2) with $1 / 2<\alpha=\beta<1$ holds:
(i) If $0 \leq c<\infty$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n\}=\frac{c+\mathbf{P}\left\{Z_{\infty}=n\right\}}{1+c} \tag{6.14}
\end{equation*}
$$

(ii) If $c=\infty$ then (6.13) is fulfilled.
4) If (A.2) holds, $\beta \geq 1$ and $m_{B}=\mathbf{E} T_{1}<\infty$, then

$$
\lim _{t \rightarrow \infty}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

Theorem 6.5. Assume (6.1), (6.2), (6.3) with $d=\infty,(6.6),(6.7)$ and (6.4). Denote by $\beta=\frac{\theta \kappa}{1+\delta}$ :

1) If $1 / 2<\beta<1$ and, either (A.1) or (A.2) with $\beta<\alpha \leq 1$ is fulfilled, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n\}=\mathbf{P}\left\{\tilde{Z}_{\infty}=n\right\}
$$

where the p.g.f. $f_{\delta, \theta}(s)=\mathbf{E} s^{\tilde{Z}_{\infty}}=1-\left(1-f_{\delta}(s)\right)^{\theta}$.
2) If (A.2) with $1 / 2<\alpha<\beta<1$ is fulfilled, then
(6.15) $\quad \lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n \mid Z(t)>0\}=\mathbf{P}\left\{\tilde{Z}_{\infty}=n\right\}$,
3) Let (A.2) with $1 / 2<\alpha=\beta<1$ holds:
(i) If $0 \leq c<\infty$, then

$$
\lim _{t \rightarrow \infty} \mathbf{P}\{Z(t)=n\}=\frac{c+\mathbf{P}\left\{\tilde{Z}_{\infty}=n\right\}}{1+c}
$$

(ii) If $c=\infty$, then (6.15) is fulfilled.
4) If (A.2) holds, $\beta \geq 1$ and $m_{B}=\mathbf{E} T_{1}<\infty$, then

$$
\lim _{t \rightarrow \infty}\{Z(t) \leq x \mid Z(t)>0\}=\frac{1}{m_{B}} \int_{0}^{\infty} \mathbf{P}\left\{Z_{1}(u) \leq x, T_{1}>u\right\} d u
$$

## REFERENCES

[1] K. K. Anderson, K. B. Athreya. A renewal theorem in the infinite mean case. Ann. Prob., 15 (1987), 388-393.
[2] S. Assmusen. Applied Probability and Queues. New York, John Wiley \& Sons, 1987.
[3] K. B. Athreya, P. Ney. A new approach to the limit theorems of recurrent Markov chains, Trans. Amer. Math. Soc., 245 (1978), 157-181.
[4] N. H. Bingham, C. M. Goldie, J. L. Teugels. Regular Variation. Cambridge, Cambridge University Press, 1987.
[5] E. B. Dynkin. Some limit theorems for sums of independent random variables with infinite mathematical expectations. Selected translations Math. Stat. and Prob., Inst. Math. Stat. Amer. Math. Soc., 1 (1961), 171-189.
[6] K. B. Erickson. Strong renewal theorems with infinite mean. Trans. Amer. Math. Soc., 151 (1970), 263-291.
[7] V. Kalashnikov, H. Thorisson. Applications of coupling and regeneration. Acta Applicandae Mathematicae, 34 (1994), Special issue.
[8] J. F. C. Kingman. Regenerative Phenomena. New York, John Wiley \& Sons, 1972.
[9] J. Lamperti. An invariance principle in renewal theory. AMS, 33 (1962), 685-696.
[10] K. V. Mitov, N. M. Yanev. Bellman-Harris branching processes with state-dependent immigration. J. Appl. Prob., 22 (1985), 757-765.
[11] K. V. Mitov, N. M. Yanev. Bellman-Harris branching processes with a special type of state-dependent immigration. Adv. Appl. Prob., 21 (1989), 270-283.
[12] K. V. Mitov, N. M. Yanev. Regenerative processes in the infinite mean cycle case. J. Appl. Prob., 38 (2001), No. 1.
[13] K. Sigman, R. W. Wolf. A review of regenerative processes. SIAM Review, 35 (1993), 269-288.
[14] W. L. Smith. Regenerative stochasti-c processes. Proc. Roy. Soc. London, Ser. A, 232 (1955), 9-48.
[15] W. L. Smith. Renewal theory and its ramifications. J. Roy. Stat. Soc. Ser. B, 20 (1958), 6-31.
[16] В. ФелЛЕР. Введение в Теорию Вероятностей и ее Приложения, т. 2. Москва, Мир, 1984.
[17] В. А. ВАтутин. Предельные теоремы для критического ветвящегося процесса Беллма-на-Харриса с бессконечной дисперсией. Теория вероятностей и ее применения, 21 (1976), 861-863.
[18] В. А. ВАтутин. Дискретные предельные расспределения числа частиц в критических ветвящеихся процессах Беллмана-Харриса. Теория вероятностей и ее применения, 22 (1977), 150-155.
[19] В. А. ВАтутин. Новая предельная теорема для критического ветвящегося процесса Беллмана-Харриса. Математический сборник, 109 (1979), 440-452.
[20] В. А. ВАтутин. Ветвящиеся процессы с бессконечной дисперсией. Сборник 4-та МЛШ по Теория на вероятностите и математическа статистика - Варна, 4 (1983), 11-38.

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# ГРАНИЧНИ ТЕОРЕМИ ЗА ПРОЦЕСИ НА ВЪЗСТАНОВЯВАНЕ, РЕГЕНЕРИРАЩИ И РАЗКЛОНЯВАЩИ СЕ ПРОЦЕСИ 

Косто Вълов Митов, Николай Михайлов Янев

В доклада са представени нови гранични теореми за един клас неотрицателни, алтернираши, регенериращи процеси и за вложените в тях процеси на възстановяване с две алтернативни състояния. Тези резултати дават нов метод за по-нататъшно изследване на процесите на Белман-Харис с имиграция, зависеща от състоянието на процеса. За тези процеси са намерени граничните разпределения в критическия случай с безкрайна дисперсията на потомството на една частица.


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