# МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2001 MATHEMATICS AND EDUCATION IN MATHEMATICS, 2001 <br> Proceedings of Thirtieth Spring Conference of the Union of Bulgarian Mathematicians <br> Borovets, April 8-11, 2001 

## SPECIAL NETS IN A THREE-DIMENSIONAL RIEMANNIAN SPACE

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With the help of the torse-forming vector fields, introduced by Yano and Yamaguchi and using Leontiev's quasidekart parallel transfer as well, this paper introduces nets in which one of the platforms is quasidekart. Symmetric nets are also considered in the paper. Characteristics of the nets and of spaces, containing such nets, are defined. Two new connectednesses, simply defined by a given net in a three-dimensional Reimannian space, are introduced. Some transformations of connectednesses, retaining properties of special nets are described.

1. Preliminary. Suppose there is a three-dimensional Reimannian space with a metric tensor $g_{i s}$. By ${ }^{1} \Gamma_{i s}^{k}$ we note the coefficients of a Levi-Chevita connectedness, corresponding to the metric tensor $g_{i s}$.

The independent unit vector fields $\underset{\alpha}{v^{i}}(\alpha=1,2,3)$ define a net $\left.\underset{1}{v} \underset{2}{v} \underset{2}{v}, \underset{3}{v}\right)$ in the space $V_{3}$. We find the mutual covectors $\stackrel{\alpha}{v}_{i}(\alpha=1,2,3)$ from the equations

$$
\begin{equation*}
\stackrel{v}{\alpha}_{v^{i} v_{s}^{\alpha}}^{=} \delta_{s}^{i}, \quad v_{\alpha}^{v^{i}}{ }_{i}^{\beta}=\delta_{\alpha}^{\beta} . \tag{1}
\end{equation*}
$$

Apart from the covectors $\stackrel{\alpha}{v}_{i}$ we also consider the covectors
(2)

$$
\underset{\alpha}{v_{i}}=g_{i k} v_{\alpha}^{k}
$$

i.e., we use $g_{i s}$ for raising and bringing down indexes.

The following equations hold:

$$
\begin{equation*}
g_{i s} v_{\alpha}^{i} v^{s}=1, \quad g_{i s} v_{\alpha}^{i} v^{s}=\cos \underset{\alpha \beta}{\omega} \tag{3}
\end{equation*}
$$

where $\underset{\alpha \beta}{\omega}$ is the angle between the vector fields $v_{\alpha}^{i}$ and $v_{\beta}^{i}$.
According to (2) the equations (3) correspondingly take the form

The covariant derivative with respect to the connectedness ${ }^{\alpha} \Gamma_{i s}^{k}$ is further noted by ${ }^{\alpha} \nabla$

The following derivative equations are obtained in [6]

$$
\begin{equation*}
{ }^{1} \nabla_{i} v_{\alpha}^{s}={\underset{\alpha}{\sigma}}_{\sigma}^{\sigma} v_{\sigma}^{s}, \quad{ }^{1} \nabla_{i}{ }^{\alpha} v_{s}=-\stackrel{-}{\sigma}_{\sigma}^{\alpha} v_{s}^{\sigma} \tag{5}
\end{equation*}
$$

Assume there is a given connectedness ${ }^{\alpha} \Gamma_{i s}^{k}, \alpha \neq 1$ in $V_{3}$. Transition from the connectedness ${ }^{1} \Gamma_{i s}^{k}$ into the connectedness ${ }^{\alpha} \Gamma_{i s}^{k}$ is considered as a transformation of ${ }^{1} \Gamma_{i s}^{k}$ into ${ }^{\alpha} \Gamma_{i s}^{k}$. According to [2, p 128], the tensor of affined deformation has the form

$$
\begin{equation*}
T_{i s}^{k}={ }^{\alpha} \Gamma_{i s}^{k}-{ }^{1} \Gamma_{i s}^{k} \tag{6}
\end{equation*}
$$

The platform $(\underset{1}{v}, \underset{2}{v})$ is quasidekart if it is transferred in quasiparallel with each of the lines. Suppose that $(\underset{1}{v}, \underset{2}{v})$ is quasidekart. According to Leontiev [1] the derivative equations (5) have the form

Yano [5] and Yamaguchi [4] call the fields ${\underset{1}{v}}^{i}$ and $\underset{2}{v^{i}}$ satisfying (7) torse-forming.

## 2. Special nets in $\boldsymbol{V}_{3}$.

2.1. Let the independent vector fields ${\underset{\alpha}{v}}_{v^{i}}(\alpha=1,2,3)$ be given in $V_{3}$. Taking into consideration (2), (3) and (5), we find

$$
\begin{equation*}
\nabla_{k} v_{\alpha}^{s}={\underset{\alpha}{\sigma}}_{\sigma}^{\sigma} v^{3} . \tag{8}
\end{equation*}
$$

Applying the integrability condition from (5), we obtain

$$
\begin{equation*}
\left.\frac{1}{2} R_{j k m} \cdot{ }_{\alpha}^{i} v^{m}=\left(\nabla_{[j} \stackrel{\nu}{T}_{\alpha}^{\nu}+\stackrel{N}{\sigma}_{\nu}^{\nu}{ }_{\alpha}^{\sigma}{ }_{\alpha}^{\sigma} k\right]\right) v_{\nu}^{i} \tag{9}
\end{equation*}
$$

After contracting (9) by $\stackrel{\alpha}{v}_{i}$ and taking into account (1) and the equation $R_{j k i}{ }^{i}=0$ we find

$$
\begin{equation*}
\nabla\left[j{\underset{\alpha}{\alpha}}_{k}^{\alpha}+{\underset{\sigma}{\alpha}}_{\alpha}^{\alpha}{\underset{\alpha}{\alpha}}_{\underset{\alpha}{\alpha}}^{k]}=0\right. \tag{10}
\end{equation*}
$$

Thus we proved
Proposition 1. The coefficients of the derivative equations (5) satisfy the equation (10).

From (9) it easy follows the validity of the
Proposition 2. The space $V_{3}$ is affined if and only if it contains a net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$, whose coefficients from the derivative formulae satisfy the condition

$$
\left.\nabla_{[j} \stackrel{\rightharpoonup}{\alpha}_{\alpha}^{\nu} k\right]+\stackrel{\rightharpoonup}{\sigma}_{\sigma}^{\nu}[j \underset{\alpha}{\underset{T}{T}} k]=0
$$

Definition. We call a net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{ }) \in V_{3}$ symmetric if $\underset{12}{\omega}=\underset{23}{\omega}=\underset{31}{\omega}$.
Assume that the net $\underset{1}{v}, \underset{2}{v}, \underset{3}{v}$ ) is symmetric and isogonal, i.e. we have for the net angles $\underset{12}{\omega}=\underset{23}{\omega}=\underset{31}{\omega}=\omega=$ const.

Then from (5) and (4) we find

$$
\begin{align*}
& \stackrel{@}{\bigoplus_{@}} k+(\underset{\alpha}{\beta} k+\underset{\alpha}{T} k) \cos \omega=0, \quad \alpha \neq \beta \neq \sigma \neq \alpha . \tag{11}
\end{align*}
$$

(The circled indexes are not to be summed)
Thus we proved the
Proposition 3. If the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_{3}$ is symmetric and isogonal, the coefficients from the derivative formulae satisfy the conditions (11).

Let the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_{3}$ is orthogonal. From (11) we find, that the coefficients of the derivative equations satisfy the equations

$$
\begin{equation*}
\stackrel{\beta}{T}_{\underset{\alpha}{*}}+\stackrel{\alpha}{T_{\beta}} k=0, \quad \stackrel{@}{T}_{\bigotimes}^{@}=0 . \tag{12}
\end{equation*}
$$

It is easy provable that if the conditions (12) are satisfied for the isogonal net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in$ $V_{3}$, then it is orthogonal as well.

Let consider the tensor

$$
\begin{equation*}
a_{i s}=\stackrel{1}{v}_{i} \stackrel{1}{v}_{s}+\stackrel{2}{v}_{i} \stackrel{2}{v}_{s}+\stackrel{3}{v_{i}} \stackrel{3}{v}_{s}, \tag{13}
\end{equation*}
$$

which is simply defined by the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$.
Assume that the equation

$$
\begin{equation*}
{ }^{2} \nabla_{k} a_{i s}=0 \tag{14}
\end{equation*}
$$

is fulfilled
From (14) we find

$$
\begin{equation*}
{ }^{2} \Gamma_{i s}^{k}=\frac{1}{2} a^{k m}\left(\partial_{i} a_{s m}+\partial_{s} a_{i m}-\partial_{m} a_{i s}\right) . \tag{15}
\end{equation*}
$$

According to (6) and (15) we find

$$
T_{i s}^{k}=\frac{1}{2} a^{k m}\left(\nabla_{i} a_{s m}+\nabla_{s} a_{i m}-\nabla_{m} a_{i s}\right)
$$

Let in the connectedness ${ }^{2} \Gamma_{i s}^{k}$ the derivative equations have the form

$$
\begin{equation*}
{ }^{2} \nabla_{k} v_{\alpha} v^{i}=\stackrel{\stackrel{\sigma}{\alpha}_{\alpha}}{k} v_{\sigma}^{v^{i}}, \quad{ }^{2} \nabla_{k} \stackrel{\alpha}{v_{i}}=-\stackrel{\alpha}{P_{k}}{ }_{k}^{\sigma} v_{i} \tag{16}
\end{equation*}
$$

From (13), (14) and (16) we establish
and after contracting with $\underset{\alpha}{v^{i} v^{s}}$ and $\underset{\alpha}{v^{i} v^{s}}$ we obtain $\stackrel{@}{@}_{\underset{@}{P}}^{k}=0$ and $\underset{\alpha}{\underset{\sim}{P}} k+\underset{\beta}{\underset{P}{P}} k=0$ respectively.
Consequently, the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$, is orthogonal in the connectedness ${ }^{2} \Gamma_{i s}^{k}$.
2.2. Suppose there is a given conforming transformation $g_{i s}=e^{2 \sigma} g_{i s}$ in $V_{3}\left(g_{i s}\right)$.
2.2.1. Let consider the connectedness [3]

$$
\begin{equation*}
{ }^{3} \Gamma_{i s}^{k}={ }^{1} \Gamma_{i s}^{k}+\sigma_{i} \delta_{s}^{k}-g_{i s} \sigma^{k}, \sigma_{i}=\partial_{i} \sigma, \sigma^{k}=g^{i k} \sigma_{i} . \tag{17}
\end{equation*}
$$

Assume that the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{2}) \in V_{3}$ is orthogonal. In accordance with (17) we have

$$
{ }^{3} \nabla_{i} v_{\alpha}^{k}={ }^{1} \nabla_{i} v_{\alpha}^{k}+\sigma_{i} v_{\alpha}^{k}-g_{i s} v_{\alpha}^{s} \sigma^{k}
$$

from where it follows

$$
\begin{equation*}
{ }_{\beta}^{v^{i 3}} \nabla_{i} v_{\alpha}^{k}=\underset{\beta}{v^{i 1}} \nabla_{i} v_{\alpha}^{k}+\sigma_{i} v_{\beta}^{i} v_{\alpha}^{k} . \tag{18}
\end{equation*}
$$

From (18) follows the validity of
Proposition 4. The orthogonal net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}) \in V_{3}$ is Chebyshevian of the first type in ${ }^{1} \Gamma_{i s}^{k}$ if and only if, it is Chebyshevian of the first type in ${ }^{3} \Gamma_{i s}^{k}$.
2.2.2. Let consider the connectedness

$$
\begin{equation*}
{ }^{4} \Gamma_{i s}^{k}={ }^{1} \Gamma_{i s}^{k}+\delta_{i}^{k} \sigma_{s}+\delta_{s}^{k} \sigma_{i}-a_{i s} \sigma^{k} . \tag{19}
\end{equation*}
$$

According to (1), (13) and (19) we have

$$
\begin{aligned}
& v^{i 4} \nabla_{i} v_{\alpha}^{k}=v_{\beta}^{i 1} \nabla_{i} v_{\alpha}^{k}+\sigma_{i} v_{\beta}^{k} v^{i}+v_{\alpha}^{k} \sigma_{i} v_{\beta}^{i} \\
& v_{\alpha}^{i 4} \nabla_{i} v_{\alpha}^{k}=v_{\alpha}^{v^{i 1}} \nabla_{i}{\underset{\alpha}{v}+v_{\alpha}^{k} \sigma_{s} v_{\alpha}^{s}-\sigma^{k}}^{\alpha}
\end{aligned}
$$

from where we obtain the validity of the following
Proposition 5. If two of the conditions below are fulfilled,

1) The field $\underset{\alpha}{v^{i}}$ is parallelly transferred along the lines $(\underset{\beta}{v})$ and $\left.\underset{\gamma}{v} \underset{\gamma}{v}\right),(\alpha \neq \beta \neq \gamma \neq \alpha)$ in the connectedness ${ }^{1} \Gamma_{i s}^{k}$;
2) the field $\underset{\alpha}{v^{i}}$ is parallelly transferred along the lines $(\underset{\beta}{v})$ and $\left.\underset{\gamma}{v}\right),(\alpha \neq \beta \neq \gamma \neq \alpha)$ in the connectedness ${ }^{4} \Gamma_{i s}^{k}$;
3) conforming transformation vector satisfies the condition $\sigma_{s}=\lambda \stackrel{\beta}{v_{s}}+\mu \stackrel{\alpha}{v_{s}}$, then the third condition is also fulfilled.

Proposition 6. If two of the following conditions are fulfilled:

1) The field $v^{i}$ is geodesic in the connectedness ${ }^{1} \Gamma_{i s}^{k}$;
2) the field $v_{\alpha}^{i}$ is geodesic in the connectedness ${ }^{4} \Gamma_{i s}^{k}$;
3) $\sigma^{k}=\lambda v_{\alpha}^{k}$, then the third condition is also satisfied.
2.3. Let the platform $(\underset{1}{v}, \underset{2}{v}$ ) $)$ be quasidekart. From (2) and (7) we find

$$
\begin{align*}
& \nabla_{k} v_{3}=\frac{T}{3} k_{1}^{v} v_{s}+\underset{3}{T} k_{2}^{2} v_{s}+\underset{3}{T} k_{3} v_{s} . \tag{20}
\end{align*}
$$

Definition. We shall call the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ a Kd- $\alpha$ net, if the platform $(\underset{\beta}{v}, \underset{\gamma}{v}),(\alpha \neq \beta \neq$ $\gamma \neq \alpha)$ is quasidekart.

The derivative equations for $\mathrm{Kd}-3$ net have the form (7) and (20).
We choose $\operatorname{Kd}-3$ net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ as a coordinate system.
From (7) we find

$$
\begin{equation*}
\Gamma_{k 1}^{3}=0, \quad \Gamma_{k 2}^{3}=0, \quad k \neq 3 \tag{21}
\end{equation*}
$$

The opposite is also easy provable, if $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}$ ) is a coordinate system and the coefficients of the connectedness ${ }^{1} \Gamma_{i s}^{k}$ satisfy (21), then $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is a Kd-3 net. Thus it follows

Proposition 7. A coordinate net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v}$ ) is a Kd-3 net if and only if the coefficients of connectedness satisfy the conditions (21).

Assume that we choose the $\operatorname{Kd}-3$ net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{2})$ as a coordinate one. From (7) for the functions $\underset{1}{A}$ and $\underset{2}{A}$ we find ${\underset{1}{1}}_{A}=\Gamma_{31}^{3}, \underset{2}{A}=\Gamma_{32}^{3}$.

Let us consider the connectednesses

$$
\begin{equation*}
{ }^{5} \Gamma_{i s}^{k}={ }^{1} \Gamma_{i s}^{k}+\delta_{i}^{k} P_{s}, \quad{ }^{6} \Gamma_{i s}^{k}={ }^{1} \Gamma_{i s}^{k}+\left(\delta_{i}^{k} \delta_{s}^{m}+\delta_{s}^{k} \delta_{i}^{m}\right) p_{m} \tag{22}
\end{equation*}
$$

where $p_{m}$ is a given vector.
According to [2, pp.149,166] the connectedness ${ }^{5} \Gamma_{i s}^{k}$ is semisymmetric, and the connectedness ${ }^{1} \Gamma_{i s}^{k}$ and ${ }^{6} \Gamma_{i s}^{k}$ are projective in terms of each other.

From (7) and (22) it follows that the fields $v_{1}^{i}$ and $v_{2}^{i}$ are torse-forming in the connectednesses ${ }^{5} \Gamma_{i s}^{k}$ and ${ }^{6} \Gamma_{i s}^{k}$, i.e. the net $(\underset{1}{v}, \underset{2}{v}, v)$ is a Kd-3 net in these connectednesses. So we obtain the validity of following

Proposition 8. If the net $(\underset{1}{v}, \underset{2}{v}, \underset{3}{v})$ is a Kd-3 net in the connectedness ${ }^{1} \Gamma_{i s}^{k}$, then it is Kd-3 net in the connectednesses ${ }^{5} \Gamma_{i s}^{k}$ and ${ }^{6} \Gamma_{i s}^{k}$ too.

## REFERENCES

[1] E. K. Leontiev. Classification of special connectednesses and compositions of multidimensional spaces. Academic news, Maths. vol. 5 1967, 40-51.
[2] A. P. Norden. Spaces of affined connectednesses, Science publishing, Moscow, 1976.
[3] P. Stavre. Connexiuni metrice semisimetrice induce de o transformare conforma de metrica. Stud si cero mat, 35, No 3 (1983), 195-204.
[4] S. Yamaguchi. On Kahlerian torse-forming vector fields. Kodai Math. Y., 2 (1979), 103-115.
[5] K. Yano. On torse-forming directions in Riemannian Spaces. Proc. Imp. Acad. Tokyo, 20, No 79 (1949).
[6] G. Zlatanov, B. Tsareva. Geometry of the nets in equiaffine Spaces. Journal of Geometry, 55 (1996), 192-201.

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## СПЕЦИАЛНИ МРЕЖИ В ТРИМЕРНО РИМАНОВО ПРОСТРАНСТВО

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Предмет на изследването са някои специални мрежи в тримерното риманово пространство - изогонални, ортогонални и симетрични. На базата на разгледаните от Яно и Ямагучи торсообразуващи векторни полета и въведеното от Леонтиев квази-паралелно пренасяне са определени характеристики на тези мрежи и на пространствата, съдържащи такива мрежи. В работата се изучават и преобразувания на свързаности запазващи свойствата на тези мрежи. Въведени са две нови свързаности, еднозначно определени от дадена мрежа в тримерно риманово пространство, като са разгледани и техни характеристики.

