# SOME UNIQUE TERNARY CONSTANT-COMPOSITION CODES 

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The uniqueness of some ternary constant-composition codes is proved by combinatorial methods.

Keywords: Enumeration of codes, constant-composition codes, ternary codes, unique codes.

1. Introduction. For all basic notions and facts about coding theory which are not introduced here we refer to [3]. All codes to be considered are ternary constantcomposition codes.

Ternary constant-composition (TCC) codes of length $n$ are codes with constant composition of "zeros", "ones" and "twos" and minimum distance $d$. Let ( $\left.n_{0}: n_{1}: n_{2}, M, d\right)$ code denote the TCC code with $n_{0}$ "zeros", $n_{1}$ "ones" and $n_{2}$ "twos" in each codeword, $M$ codewords and minimum distance $d$. Let $A_{3}\left(n_{0}: n_{1}: n_{2}, d\right)$ denote the largest value of the size of the code $M$ such that there exists an $\left(n_{0}: n_{1}: n_{2}, M, d\right)$ code. Codes with such parameters are called optimal.

The fundamental question in coding theory is the existence of codes with given parameters. In cases when the existence problem has already been solved, the problem for the classification of all inequivalent codes with these parameters arises.

The problem of finding values of $A_{3}\left(n_{0}: n_{1}: n_{2}, d\right)$ is considered in [1], [4], [5], [2].
In this paper some TCC codes are enumerated up to equivalence. We obtain that some optimal TCC codes are unique. A family of unique TCC codes with parameters $(t+\lambda: 1: t-1,\lfloor(2 t+\lambda) / t\rfloor, 2 t)$ for every integer $t \geq 2, \lambda \geq 0$ is found.
2. Enumeration of TCC codes by combinatorial constructions. Combinatorial and computer methods can be used to classify optimal codes. To determine equivalence of TCC codes we can apply computer search to cases that we were not able to settle using the combinatorial constructions. In this paper we only present the results about unique TCC codes which are obtained by combinatorial methods.

The following notations and definitions are used in this section. Let $C$ be an $\left(n_{0}: n_{1}\right.$ : $\left.n_{2}, M, d\right)$ TCC code. Considering any position we denote by $m_{j}$ the number of codewords of value $k$ in this position, $k=0,1,3$. The codeword matrix is the $M \times n$ matrix, its rows being the codewords of $C$. Denote by $r_{i}$ the $i$-th row, and by $r_{i j}$ - the $j$-th entry of the $i$-th row.

We denote by $C_{j}^{k}$ the code $C$ shortened with respect to the symbol k in position $j$. This code is obtained by selecting a column $j$ in the codeword matrix, and considering the subcode consisting of the codewords that have the symbol k in this column $j$. This subcode with position $j$ removed is the shortened code $C_{j}^{k}$.

Let $X_{1}, X_{2}, \ldots, X_{s}$ be codes of equal lengths. We denote by $d\left(X_{1}, X_{2}, \ldots, X_{s}\right)$ the minimum distance of a code obtained from the union of the codes $X_{1}, X_{2}, \ldots, X_{s}$.

We may assume that the rows $r_{1}, r_{2}, \ldots, r_{M}$ are lexicographically ordered. The same is valid for the columns. For all codes to be considered we may assume without loss of generality (wlog) that the first row is $\underbrace{0 \ldots 0}_{n_{0}} \underbrace{1 \ldots 1}_{n_{1}} \underbrace{2 \ldots 2}_{n_{2}}$.

Definition 1. Two q-ary codes are called equivalent if one of them can be obtained from the other by a combination of operations of the following types:
A) permutation of the coordinates of the code;
B) permutation of the symbols appearing in a fixed position.

The uniqueness of some TCC codes is proved in the following theorems.
Theorem 1. There exists a family of unique up to equivalence TCC codes with parameters $(t+\lambda: 1: t-1,\lfloor(2 t+\lambda) / t\rfloor, 2 t)$ for every integer $t \geq 2, \lambda \geq 0$.

Proof. Let $C$ be a $(t+\lambda: 1: t-1, M, 2 t)$ code and the first row is $\underbrace{0 \ldots 0}_{t+\lambda} 1 \underbrace{2 \ldots 2}_{t-1}$.
Let $m_{2}=2$ in position $j$. Then the subcode $C_{j}^{2}$ is a $(t+\lambda: 1: t-2,2,2 t-1)$ code and $d<2 t$. Similarly if $m_{1}=2$ or $m_{1}+m_{2}=2$ in position $j$ then $d<2 t$. So, we have $m_{1}+m_{2}=1$ and $m_{0}=\lfloor(2 t+\lambda) / t\rfloor-1$ in all positions.

The positions of "ones" and "twos" in the next codewords must be the first $t+\lambda$ positions (the positions of "zeros" in the first codeword). Then wlog we may assume that
$r_{2}$ is $\underbrace{0 \ldots 0}_{t+\lambda-t} 1 \underbrace{2 \ldots 2}_{t-1} \underbrace{0 \ldots 0}_{t}, r_{3}$ is $\underbrace{0 \ldots 0}_{t+\lambda-2 t} 1 \underbrace{2 \ldots 2}_{t-1} \underbrace{0 \ldots 0}_{2 t}, \ldots, r_{M}$ is $\underbrace{0 \ldots 0}_{t+\lambda-M t)} 1 \underbrace{2 \ldots 2}_{t-1} \underbrace{0 \ldots 0}_{M t}$.
The rows $r_{1}, r_{2}, \ldots, r_{M}$ are uniquely determined up to equivalence. The size of the obtained code is $M=n_{0}+n_{1}+n_{2}=\lfloor(2 t+\lambda) / t\rfloor$. The code is optimal because if $M=n_{0}+n_{1}+n_{2}+1$ then $m_{1}+m_{2}>1$ and $d\left(r_{M+1}, C\right)<2 t$.

So, the $(t+\lambda: 1: t-1,\lfloor(2 t+\lambda) / t\rfloor, 2 t)$ codes are optimal unique codes. The codes can be constructed by a cyclic shift of the first codeword $\underbrace{0 \ldots 0} 1 \underbrace{2 \ldots 2}$ at $t$ positions.

Theorem 2. There exists unique up to equivalence TCC code with parameters $(2: 2: 2,15,4)$.

Proof. Let $C$ be a $(2: 2: 2,15,4)$ code. Let $m_{0}=6$ ( or $m_{1}=6$ or $m_{2}=6$ ) in position $j$. Then the subcode $C_{j}^{0}$ is a $(1: 2: 2,6,4)$ TCC code. It is proved $A(1: 2: 2,4)=5[5]$. So, the $(1: 2: 2,6,4)$ code does not exist and the subcode $C_{j}^{0}$ is equivalent to an (1:2:2,5,4) TCC code.

Then we have $m_{0}=m_{1}=m_{2}=5$ in all positions.
It is easy to prove by combinatorial methods that the $(1: 2: 2,5,4)$ TCC code is unique and the code is equivalent to the following code:

$$
\begin{array}{lllll}
0 & 1 & 1 & 2 & 2 \\
1 & 0 & 2 & 1 & 2 \\
1 & 2 & 0 & 2 & 1 \\
2 & 1 & 2 & 1 & 0 \\
2 & 2 & 1 & 0 & 1
\end{array}
$$

We may assume wlog that the first column is $(0,0,0,0,0,1,1,1,1,1,2,2,2,2,2)^{T}$ and the first five rows are:

$$
\begin{array}{llllll}
0 & 0 & 1 & 1 & 2 & 2 \\
0 & 1 & 0 & 2 & 1 & 2 \\
0 & 1 & 2 & 0 & 2 & 1 \\
0 & 2 & 1 & 2 & 1 & 0 \\
0 & 2 & 2 & 1 & 0 & 1
\end{array}
$$

The code $C_{2}^{0}$ is equivalent to a unique $(1: 2: 2,5,4) \mathrm{TCC}$ code and can be obtained from the code $C_{1}^{0}$ by the permutations given in Definition 1 . We can apply only the permutations which don't change the fixed numbers $n_{0}, n_{1}, n_{2}$ of the code $C$. For instance, if we apply the permutation $(0,1)(2)$ over the elements of columns we change the numbers $n_{0}=3$ and $n_{1}=1$. But all permutations in this case which save $n_{0}, n_{1}, n_{2}$ lead only to permutations of the coordinates of the code. For instance, the permutation $(0)(1,2)$ over the elements of all coordinates only rearranges the columns and the rows. So, the codeword matrix of $C_{2}^{0}$ differs from other codeword matrix of the $(1: 2: 2,5,4)$ only in the arrangement of the columns (and rows). Thus, we may assume wlog that the first column of the codeword matrix of $C_{2}^{0}$ is $(0,1,1,2,2)^{T}$ and that its first row is: 01122 .

From uniqueness of the $(1: 2: 2,5,4)$ code it follows that the second and third columns are one of: 10212 or 12021 . Also the forth and fifth columns are one of: 22110 or 21201 . Since $n_{0}=n_{1}=n_{2}=2$ and $d\left(r_{1}, r_{2}, \ldots, r_{6}\right)=4 C_{2}^{0}$ is uniquely determined.

Analogously we obtain the shortened subcodes $C_{3}^{0}, C_{4}^{0}, C_{5}^{0}, C_{6}^{0}$.
The codeword matrices of these subcodes are:

| $C_{2}^{0}$ | $C_{3}^{0}$ | $C_{4}^{0}$ | $C_{5}^{0}$ | $C_{6}^{0}$ |
| :---: | :---: | :---: | :---: | :---: |
| 01122 | 01212 | 01221 | 02121 | 02211 |
| 10221 | 10221. | 10212. | 11220 | 11220 |
| 12012 | 12102 | 12120 | 12012. | 12102 |
| 21210 | 21120 | 21102 | 20211. | 20121. |
| 22101 | 22011 | 22011. | 21102. | 21012. |

The rows denoted by "." are repeated. From these subcodes we obtain the remaining codewords of the code $C$.

The code $C$ is:

```
001122
010212
012021
021201
022110
100221
102012
112200
120102
121020
201210
202101
210120
211002
220011
```

Analogously the uniqueness of other cases can be proved in a similar way (by combinatorial constructions). For instance, we obtain that the optimal TCC codes with parameters $(2: 1: 1,4,3),(3: 1: 1,5,3),(2: 1: 2,10,3),(2: 1: 2,2,5),(3: 1: 2,3,5),(2: 1: 3,7,5)$, $(4: 1: 3,4,6),(5: 1: 3,6,6),(4: 1: 4,9,6),(4: 1: 4,2,9),(4: 2: 3,2,9)$ and $(5: 1: 4,2,10)$ are unique up to equivalence. The uniqueness of the TCC code with parameters $(1: 1: 1,3,3)$ is proved by graphs in [5].

## REFERENCES

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## НЯКОИ ЕДИНСТВЕНИ КОНСТАТНО-КОМПОЗИЦИОННИ КОДОВЕ

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Изследван е проблема за класификация на константно-композиционни кодове. Чрез комбинаторни методи е доказана единствеността на някои констант-но-композиционни кодове.

