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AN IMPLICIT RUNGE-KUTTA METHOD FOR A CLASS OF DIFFERENTIAL INCLUSIONS: LOCAL ERROR ESTIMATES

Radostin Petrov Ivanov,
Nedka Velikova Pulova, Vladimir Ivanov Pulov

A multivalued version of an implicit Runge-Kutta method known from the field of differential equations as the implicit midpoint rule is considered. The method is applied to a class of differential inclusions. Certain local error estimates are obtained.

Introduction. We consider the initial value problem of finding an absolutely continuous function $x(t) \in \mathbb{R}^n$ on the interval $[0,T]$ satisfying the differential inclusion

$$\frac{dx(t)}{dt} \in \mathrm{co} \bigcup_{i=1}^{k} a_i(x(t)) \text{ for almost all } t \in [0,T],\ x(0) = x_0, \tag{1}$$

where co means the convex hull and the given functions $a_i(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, k$ are supposed to satisfy the following assumptions:

A(i): $a_i(\cdot)$ are twice continuously differentiable, and

A(ii): $a_i(\cdot)$ are with linear growth: $\|a_i(x)\| \leq \vartheta(1 + \|x\|)$ for some positive ϑ, where $\|\cdot\|$ is the Euclidean norm.

For any sequence of k elements $(f_1, f_2, \ldots, f_k)$ we use the notation $\{f_i\}_{i=1}^k$. Denote $\mathcal{A} = \left\{ \{\alpha_i\}_{i=1}^k \middle| \alpha_i \in R,\ \sum\limits_{i=1}^k \alpha_i = 1,\ \alpha_i \geq 0 \right\}$. The set of all measurable (single-valued) selections of the mapping $t \mapsto \mathcal{A}$, defined on $[0,T]$ is given by

$$\mathcal{M}[0,T] = \left\{ \{\mu_i(\cdot)\}_{i=1}^k \middle| \ \mu_i(\cdot) \in L_1[0,T],\ \sum_{i=1}^k \mu_i(t) = 1,\ \mu_i(t) \geq 0 \right\}.$$

Let $X[0,T]$ denotes the solutions set of (1). Thanks to the conditions A(i) and A(ii) and the lemma of Filippov in [1] it holds the following

Proposition 1.

$$X[0,T] = \left\{ x(\cdot) \middle| x(t) = x_0 + \int_0^t \sum_{i=1}^k \mu_i(s) a_i(x(s)) ds \ \text{on } [0,T], \{\mu_i(\cdot)\}_{i=1}^k \in \mathcal{M}[0,T] \right\} \tag{2}$$

Implicit midpoint rule: *For a given integer number $j \geq 0$, the initial value x_0 of the problem (1) and $h > 0$, define the vector $x_{(j+1)h}$ by*

$$x_{(j+1)h} \in \{x_{jh} + hz | z \in Z(h, x_{jh})\} \equiv G_j(h, x_{jh}), \tag{3}$$

where $Z(h, x_{jh})$ denotes the solutions set of the implicit inclusion

$$z \in \left\{ \sum_{i=1}^{k} \alpha_i a_i \left(x_{jh} + \frac{h}{2} z \right) \middle| \{\alpha_i\}_{i=1}^{k} \in \mathcal{A} \right\}. \tag{4}$$

Given an uniform grid $0 = t_0 < t_1 < \ldots < t_m = T$ with stepsize $h = \dfrac{T}{m}$, we consider the approximate solution of (1) to be any sequence $\mathcal{X}_h = (x_0, x_h, \ldots, x_{mh})$ obtained by (3) and (4). The reachable set of (1) at the point t in $[0, T]$ is $R(t) = \{x(t) | x(\cdot) \in X[0, T]\}$. The reachable set associated with the discretezation (3)-(4) is $R_{jh} = \{x | x = x_{jh}$ for some $\mathcal{X}_h\}$.

The proofs of the next two assertions are standard and will be omitted.

Lemma 1. *Suppose $f : R \to R^n$ is continuous function satisfying the condition of linear growth $\|f(x)\| \leq \vartheta(1 + \|x\|)$ for some $\vartheta > 0$. If $\eta > 0$ be given, then for $\|x\| \leq \eta$ and positive $h \leq h_0 < \dfrac{2}{\vartheta}$, there exists $z(h, x) \in \mathbb{R}^n$ such that:*

$$z(h, x) = f(x + \frac{h}{2} z(h, x)) \text{ and } \|z(h, x)\| \leq \frac{2\vartheta}{2 - \vartheta h_0}(1 + \eta).$$

Proposition 2. *Let $T > 0$ be given. Consider the implicit midpoint rule with $h = \dfrac{T}{m}$ for some integer $m > 0$. Assume $a_i(\cdot)$, $i = 1, 2, \ldots, k$ are continuous functions satisfying the condition of linear growth A(ii), and denote $C = \dfrac{2\vartheta}{2 - \vartheta h_0}$. Then for each $\{\alpha_i\}_{i=1}^{k} \in \mathcal{A}$ and $h \leq h_0 < \dfrac{2}{\vartheta}$ there exist $z(h, x_{jh}) \in Z(h, x_{jh})$, $j = 0, 1 \ldots, m - 1$ such that*

$$\max_{j=0,1,\ldots,m-1} \|z(h, x_{jh})\| < C(1 + \|x_0\|)e^{CT} \text{ and } \max_{j=0,1,\ldots,m} \|x_{jh}\| < (1 + \|x_0\|)e^{CT} - 1.$$

Let B denotes some closed ball in $\mathbb{R}^n$ that contains the values of all exact solutions in $X[0, T]$ and the values x_{jh} of all approximate solutions in $X_h[0, T]$ for $h < \dfrac{2}{\vartheta}$. Via the assumption A(i) we take advantage of the obvious Lipschitz continuity of the given $a_i(\cdot)$ on B: $\|a_i(x) - a_i(y)\| \leq M_1\|x - y\|$ for $x \in B$, $y \in B$.

Main result.

Lemma 2. *Consider the family of sequences*

$$\mathcal{L}^{(l)}[0, T] = \left\{ \left\{ \lambda_i^{(l)}(\cdot) \right\}_{i=1}^{k} \middle| \lambda_i^{(l)}(\cdot) \in C^l[0, T],\ \sum_{i=1}^{k} \lambda_i^{(l)}(t) = 1,\ \lambda_i^{(l)}(t) \geq 0 \right\},$$

where $C^l[0,T]$, $l = 0, 1, 2, \dots$ is the space of l-times continuously differentiable functions on $[0,T]$. We write $C[0,T] \equiv C^0[0,T]$, $\mathcal{L}[0,T] \equiv \mathcal{L}^{(0)}[0,T]$, $\lambda_i(\cdot) \equiv \lambda_i^{(0)}(\cdot)$, etc. Denote

$$(5) \quad \widetilde{X}^{(l)}[0,T] =$$

$$\left\{ \widetilde{x}^{(l)}(\cdot) | \widetilde{x}^{(l)}(t) = x_0 + \int_0^t \sum_{i=1}^k \lambda_i^{(l)}(s) a_i(\widetilde{x}^{(l)}(s))\, ds \text{ on } [0,T],\ \{\lambda_i^{(l)}(\cdot)\}_{i=1}^k \in \mathcal{L}^{(l)}[0,T] \right\},$$

$$\widetilde{R}^{(l)}(t) = \left\{ \widetilde{x}^{(l)}(t) | \widetilde{x}^{(l)}(\cdot) \in \widetilde{X}^{(l)}[0,T] \right\}.$$

We claim that, on the assumptions A(i) and A(ii), $\widetilde{X}^{(l)}[0,T]$ and $\widetilde{R}^{(l)}(t)$ are dense subsets in $X[0,T]$ and $R(t)$:

$$X[0,T] = cl\, \widetilde{X}^{(l)}[0,T] \text{ in the uniform metric of } C[0,T],$$

$$R(t) = cl\, \widetilde{R}^{(l)}(t) \text{ in the norm metric of } \mathbb{R}^n, \text{ uniformly in } t,$$

where $cl\, A$ is the closure of the set $A \subset \mathbb{R}^n$.

Proof. Let us first recall the well known statement: for any function $\mu(\cdot) \in L_1[0,T]$ and for any $\varepsilon > 0$, there exists $\tau > 0$, such that the Steklov's mean function $\lambda : [0,T] \to \mathbb{R}$, defined by

$$\lambda(t) = \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} \nu(s)\, ds,\ \nu(s) = \{\mu(s) \text{ if } t \in [0,T] \text{ and } 0 \text{ if } t \in \mathbb{R} \setminus [0,T]\}$$

(see e.g. in [2]) is continuous and $\|\mu(\cdot) - \lambda(\cdot)\|_{L_1} < \varepsilon$; $\|\cdot\|_{L_1}$ is the norm in $L_1[0,T]$. Denote by M the maximum of the Lipschitz constant M_1 and the upper bounds of $\|a_i(\cdot)\|$, $i = 1, \cdots, k$ on B. By the chain of inequalities

$$\begin{aligned}
&\|x(t) - \widetilde{x}^{(l)}(t)\| = \left\| \sum_{i=1}^k \int_0^t (\mu_i(s) a_i(x(s)) - \lambda_i^{(l)}(s) a_i(\widetilde{x}^{(l)}(s)))\, ds \right\| \\
&\le \sum_{i=1}^k \int_0^t |\mu_i(s) - \lambda_i^{(l)}(s)| \|a_i(x(s))\|\, ds + \sum_{i=1}^k \int_0^t \lambda_i^{(l)}(s) \|a_i(x(s)) - a_i(\widetilde{x}^{(l)}(s))\|\, ds \\
&\le kM\varepsilon + M \int_0^t \|x(s) - \widetilde{x}^{(l)}(s)\|\, ds \le kM\varepsilon + \int_0^t \varepsilon k M^2 e^{\int_0^t M\, d\xi}\, ds \le kM(1 + MTe^{MT})\varepsilon
\end{aligned}$$

we complete the proof. $\square$

Theorem. *If the conditions* A(i) *and* A(ii) *are assumed, then for sufficiently small h the following estimates hold true:*

1. *For any x_h in R_h, there exists a $x(h)$ in $R(h)$ such that $\|x(h) - x_h\| \le \mathcal{O}(h^3)$, where $\mathcal{O}(h^3)$ does not depend on x_h;*

2. *For a given solution $x(\cdot) \in X[0,T]$, there exists $x_h \in R_h$ such that $\|x(h) - x_h\| = o(h^2)$, where $o(h^2)$ depends on $x(\cdot)$.*

Proof. By applying the Taylor's formula to the functions $a_i(\cdot)$ we obtain

$$\begin{aligned} x(h) - x_h &= \int_0^h \sum_{i=1}^k \left[\mu_i(s) a_i(x(s)) - \alpha_i a_i(x_0 + \frac{h}{2} z(h, x_0))\right] ds \\ &= \int_0^h \sum_{i=1}^k (\mu_i(s) - \alpha_i) a_i(x_0)\, ds + \int_0^h \sum_{i=1}^k \mu_i(s) a_i'(x_0)(x(s) - x_0) \\ &\quad - \int_0^h \sum_{i=1}^k \alpha_i a_i'(x_0) s z(h, x_0)\, ds + \mathcal{O}(h^3). \end{aligned}$$

The term $\mathcal{O}(h^3)$ depends on the Lipschitz constant of the solutions of (1), the upper bound of the second derivatives $\left|\dfrac{\partial^2 a_{i,r}}{\partial x_p \partial x_q}\right|$ of the vector components $a_{i,r}$, $r = 1, 2, \ldots, n$ on B (implied by A(i)), and the upper bound of $\|z(h, x_0\|$ for sufficiently small h (see the Proposition 2). Interpolating by $x_s = x_{jh} + (s - jh) z(h, x_{jh})$, and applying the Gronwall's inequality, we obtain ($M_0 = \sum_{i=1}^k \|a_i'(x_0)\|$)

$$\|x(h) - x_h\| \le c_1(h) + c_2(h) + \mathcal{O}(h^3) + M_0 e^{M_0 h} \int_0^h (c_1(h) + c_2(h) + \mathcal{O}(h^3))\, ds, \tag{6}$$

where

$$c_1(h) = \left\| \sum_{i=1}^k a_i(x_0) \int_0^h (\mu_i(s) - \alpha_i)\, ds \right\| \tag{7}$$

$$c_2(h) = \|z(h, x_0)\| \sum_{i=1}^k \|a_i'(x_0)\| \left| \int_0^h (\mu_i(s) - \alpha_i) s\, ds \right|. \tag{8}$$

1. Let us fix an element in R_h associated with the sequence $\{\alpha_i\}_{i=1}^k$ in $\mathcal{A}$. Hence, by the sequence of functions $\{\mu_i(\cdot)\}_{i=1}^k$ in $\mathcal{M}[0, T]$ defined on $[0, h]$ by the substitutions $\mu_i(s) = \alpha_i$, we choose a point $x(h)$ in $R(h)$, for which the inequality (6) implies the desired estimate.

2. We shall first consider the set $\widetilde{X}[0, T]$. Let us fix a sequence $\{\lambda_i(\cdot)\}_{i=1}^k$ in $\mathcal{L}[0, T]$, that is equivalent to choose a solution $\widetilde{x}(\cdot)$ in $\widetilde{X}[0, T]$. Then, we put $\lambda_i(\cdot)$ for the functions $\mu_i(\cdot)$ into equations (7) and (8), and set $\alpha_i(h) = \dfrac{1}{h} \int_0^h \lambda_i(s)\, ds$, $i = 1, 2, \cdots, k$. It is easy

to check that $\left|\int_0^h (s-\frac{h}{2})\lambda_i(s)\,ds\right| \le \frac{h^2}{8}$, and hence, applying the L'Hopital rule we have

$$\lim_{h\to 0}\frac{1}{h^2}\int_0^h \left(s-\frac{h}{2}\right)\lambda_i(s)\,ds = \lim_{h\to 0}\frac{1}{2h}\left(\int_0^h \left(-\frac{1}{2}\right)\lambda_i(s)\,ds + \frac{h}{2}\lambda_i(h)\right)$$
$$= -\lim_{h\to 0}\frac{1}{4h}\int_0^h \lambda_i(s)\,ds + \frac{1}{4}\lim_{h\to 0}\lambda_i(h) = -\frac{1}{4}\lim_{h\to 0}\lambda_i(h) + \frac{1}{4}\lambda_i(0) = 0.$$

As a result we obtain $c_1(h) = 0$ and $c_2(h) = o(h^2)$. For the latter estimate we make use of the boundedness of $\|z(h, x_0)\|$ (see the Proposition 2). Therefore, it follows from (6) that for a given solution $\widetilde{x}(\cdot) \in \widetilde{X}[0,T]$, the defined sequence $\{\alpha_i(h)\}_{i=1}^k \in \mathcal{A}$ determines a point $x_h \in R_h$ which satisfies $\lim_{h\to 0}\frac{1}{h^2}\|\widetilde{x}(h) - x_h\| = 0$. In the remainder of the proof we fix an arbitrary $x(\cdot)$ in $X[0,T]$. From the Lemma 2 we have that for any $\varepsilon > 0$, there exists $\widetilde{x}(\cdot) \in \widetilde{X}[0,T]$, such that

$$\frac{\|x(h) - x_h\|}{h^2} \le \frac{\|x(h) - \widetilde{x}(h)\|}{h^2} + \frac{\|\widetilde{x}(h) - x_h\|}{h^2} < \frac{\varepsilon}{h^2} + \frac{\|\widetilde{x}(h) - x_h\|}{h^2}$$

holds for any h in $[0,T]$. Hence, by choosing $\varepsilon = o(h^2)$ we complete the proof. $\square$

To recall, the Hausdorff distance between the sets A and B is $\mathrm{haus}(A,B) = \max\{H(A,B), H(B,A)\}$, where $H(A,B) = \sup_{x\in A}\inf_{y\in B}\|y - x\|$.

Corollary. *On the assumptions* A(i) *and* A(ii) *for sufficiently small* h *it holds*

$$haus(R_h, R(h)) \le \mathcal{O}(h^2).$$

Proof. It suffices to invoke the substitutions $\alpha_i(h) = \frac{1}{h}\int_0^h \mu_i(s)\,ds$, $i = 1, 2, \cdots, k$ to deduce from (6): for every $x(\cdot) \in X[0,T]$ there exists $x_h \in R_h$, such that $\|x(h) - x_h\| \le \mathcal{O}(h^2)$, where $\mathcal{O}(h^2)$ does not depend on $x(\cdot)$. On the other hand, from the first statement of the Theorem it follows $H(R_h, R(h)) \le \mathcal{O}(h^3)$. The last two estimates imply the claim. $\square$

Remark. The uniformity of the second estimate of the Theorem in respect with the solutions $x(\cdot)$ in $X[0,T]$ should be enough to prove $o(h^2)$ order of convergence for the Hausdorff semidistance $H(R(h), R_h)$, which together with $H(R_h, R(h)) \le \mathcal{O}(h^3)$ should imply $\mathrm{haus}(R_h, R(h)) = o(h^2)$. But, it can be shown that this uniformity is not available in general and the estimate of the Hausdorff distance between R_h and $R(h)$ can not be better than $\mathcal{O}(h^2)$.

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Nedka Velikova Pulova
Department of Mathematics
TU-Varna
1, Studentska Str.
9010 Varna, Bulgaria

Vladimir Ivanov Pulov
Department of Physics
TU-Varna
1, Studentska str.
9010 Varna, Bulgaria

ЕДНА НЕЯВНА СХЕМА НА РУНГЕ-КУТА ЗА КЛАС ДИФЕРЕНЦИАЛНИ ВКЛЮЧВАНИЯ: ОЦЕНКИ НА ЛОКАЛНАТА ГРЕШКА

Радостин Петров Иванов,
Недка Великова Пулова, Владимир Иванов Пулов

Разгледана е многозначна версия на една неявна схема на Рунге-Кута, известна от теорията на диференциалните уравнения като неявно правило на средната точка, в приложение към клас от диференциални включвания. Получени са оценки на локалната грешка.