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ON THE NUMBER OF SOME k-VALUED FUNCTIONS OF n VARIABLES*

Dimiter St. Kovachev

Let M and R be sets of variables of the function $f(x_1, x_2, \ldots, x_n) \in P_n^k$, where P_n^k is the set of all k-valued functions of n variables. In this article the number of functions $f \in P_n^k$ is found, for which

- the set M is **separable** for f;
- the set M is **c-separable** for f;
- each subfunction of f with variables from the set M takes all the values of the function;
- the set M with respect to R for the function f has a given **spectrum**;
- the set M with respect to R for the function f has a given **c-spectrum** etc.

The method of counting could be used for "construction" of the considered functions, as well.

Let $P_n^k = \{f : A^n \to A/A = \{0, 1, \ldots, k-1\}\}$ be the set of all k-valued functions of n variables.

Definition 1. **([2])** *The number of different values of f, is called **range** of f.*

We will denote the **range** of function f by $\mathbf{Rng}(f)$.

Let $X_f = \{x_1, x_2, \ldots, x_n\}$ for $f(x_1, x_2, \ldots, x_n) \in P_n^k$.

Let $\lambda_n = |A^n| = k^n$ be the number of all tuples of constants for the variables of $f(x_1, x_2, \ldots, x_n) \in P_n^k$, and $\mu_n = |P_n^k| = k^\lambda = k^{k^n}$ be the number of all functions from P_n^k.

Let $q \in \{1, 2, \ldots, k\}$ and $\mu_n^k(q)$ be the number of functions from P_n^k with **range q.**

Using [1] and [2], for $\mu_n^k(q)$ we have

$$\mu_n^k(q) = C_k^q . \sum_{\substack{r_1+r_2+\ldots+r_q=k^n \\ r_i \geq 1,\ i=1,2,\ldots,q}} \frac{k^n!}{r_1!r_2!\ldots r_q!} \tag{1}$$

or

$$\mu_n^k(q) = C_k^q \sum_{s=1}^{q} (-1)^{q-s} C_q^s s^{k^n}. \tag{2}$$

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Let

(3) $$\mu_n^k(0) = 0.$$

Definition 2. *A function g is called subfunction of $f \in P_n^k$ with respect to M if g is obtained from f by replacing the variables in the set $M \subset X_f$ with constants, and we write*

$$g \overset{M}{\prec} f.$$

Definition 3. **([2])** *Let $M \subset X_f$ be a set of variables and g be a subfunction of f with respect to the set $X_f \setminus M$. The* ***range*** *of g is called* ***range*** *of M for f with respect to g, which is denoted by $\boldsymbol{Rng}(M, f; g)$, and*

$$\boldsymbol{Rng}(M, f; g) = \boldsymbol{Rng}(g).$$

Let G be the set of all subfunctions of f with respect to $X_f \setminus M$, i.e. $G = \{g : g \overset{X_f \setminus M}{\prec} f\}$.

Definition 4. **([2])** *The set $\boldsymbol{Spr}(M, f) = \cup_{g \in G}\{\boldsymbol{Rng}(M, f; g)\} = \cup_{g \in G}\{\boldsymbol{Rng}(g)\}$ is called spectrum of the set M with respect to f.*

Obviously $\mathbf{Spr}(M, f) \subseteq \{1, 2, \ldots, k\}$.

Definition 5. **([2])** max $\boldsymbol{Spr}(M, f)$ *is called* ***range*** *of M with respect to f.*

By $\mathbf{Rng}(M, f)$ we denote the **range** of M for f as

$$\mathbf{Rng}(M, f) = \max \mathbf{Spr}(M, f) = \max_{g \in G}\{\mathbf{Rng}(M, f; g)\} = \max_{g \in G}\{\mathbf{Rng}(g)\}.$$

Let $\mathbf{Q}_{n,k}$ be the number of functions from P_n^k, which have a property $\mathbf{Q}$, i.e. $\mathbf{Q}$ fully describes the $\mathbf{Q}_{n,k}$.

Theorem 1. *If $M \subset X_f$, $|M| = m > 0$ then the number of functions $f \in P_n^k$, for which*

1.1) *each subfunction $g \overset{X_f \setminus M}{\prec} f$ has the property $\boldsymbol{Q}$ is*

(4) $$[\boldsymbol{Q}_{m,k}]^{k^{n-m}};$$

1.2) *there is a subfunction $g \overset{X_f \setminus M}{\prec} f$, which has the property $\boldsymbol{Q}$ is*

(5) $$k^{k^n} - [k^{k^m} - \boldsymbol{Q}_{m,k}]^{k^{n-m}}.$$

Proof 1.1. Let $X_f \setminus M = \{x_{j_1}, x_{j_2}, \ldots, x_{j_{n-m}}\}$. Let us denote all possible tuples of constants for the variables from $X_f \setminus M$ by $\{c_1^i, c_2^i, \ldots, c_{n-m}^i\}$, $i = 1, 2, \ldots, k^{n-m}$. If $g_i = f(x_{j_1} = c_1^i, x_{j_2} = c_2^i, \ldots, x_{j_{n-m}} = c_{n-m}^i)$, then $g_i \in P_m^k$, $i = 1, 2, \ldots, k^{n-m}$.

The number of functions from P_m^k, which have a property $\mathbf{Q}$, is $\mathbf{Q}_{m,k}$. The tabular presentation of f can be viewed as k^{n-m} tables with m rows, which are the tabular presentations of the functions g_i, $i = 1, 2, \ldots, k^{n-m}$.

In view of the fact that every subfunction g_i must have the property $\mathbf{Q}$ and can be chosen among $\mathbf{Q}_{m,k}$, then the number of different functions f, for which every subfunction $g \overset{X_f \setminus M}{\prec} f$ has the property $\mathbf{Q}$ is

$$[\mathbf{Q}_{m,k}]^{k^{n-m}}.$$

Proof 1.2. Let us denote by $\mathbf{T}_{n,k} = k^{k^n} - \mathbf{Q}_{n,k}$ the number of functions from P_n^k, which do not have the property $\mathbf{Q}$. From T1, (4) it follows that the number of functions $f \in P_n^k$, for which each subfunction $g \overset{X_f \setminus M}{\prec} f$ does not have the property $\mathbf{Q}$, is

$$[\mathbf{T}_{m,k}]^{k^{n-m}} = [k^{k^m} - \mathbf{Q}_{m,k}]^{k^{n-m}}.$$

It is evident that $k^{k^n} - [k^{k^m} - \mathbf{Q}_{m,k}]^{k^{n-m}}$ is the number of functions $f \in P_n^k$, for which there is at least one subfunction $g \overset{X_f \setminus M}{\prec} f$ with the property $\mathbf{Q}$.

If $\mathbf{Q}$ is exactly the property of functions f for which $\mathbf{Rng}\ (f) = q$, then from $\mathbf{Q}_{n,k} = \mu_n^k(q)$ and Theorem 1, (4) we obtain Theorem 2.3 from [2]:

Corollary 1. *If $M \subset X_f$, $|M| = m > 0$ then the number of functions $f \in P_n^k$ for which* ***Rng***$(M, f; g) = q$, *$q \leq k$, for all $g \overset{X_f \setminus M}{\prec} f$ is*

$$\left[\mu_m^k\,(q)\right]^{k^{n-m}}. \tag{6}$$

Definition 6. **([3])** *A variable x_i is called* ***essential*** *for the function $f \in P_n^k$ if there exist values $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n$ such that the function $f(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_n)$ assumes at least two different values.*

Definition 7. **([3])** *A set M of variables is called separable for the function f if there exist a subfunction $g \overset{X_f \setminus M}{\prec} f$ which depends on all variables from M.*

[4] Taking into consideration that $\mathbf{Q}_{n,k} = \displaystyle\sum_{j=0}^{n} (-1)^j C_n^j k^{k^{n-j}}$ is the number of functions from P_n^k which depend essentially on all their variables (property $\mathbf{Q}$**),** Definition 7 and Theorem 1, (5) we obtain

Corollary 2. *If $M \subset X_f$, $|M| = m > 0$, then the number of functions $f \in P_n^k$ for which M is* ***separable*** *is*

$$k^{k^n} - \left[\sum_{j=1}^{m} (-1)^{j+1} C_m^j k^{k^{m-j}}\right]^{k^{n-m}}. \tag{7}$$

Definition 8. **([3])** *A set M of essential variables for f is called* ***c-separable*** *if each subfunction $g \overset{X_f \setminus M}{\prec} f$ depends on the variables in M.*

Corollary 3. *If $M \subset X_f$, $|M| = m > 0$, then the number of functions $f \in P_n^k$ for which M is* ***c-separable*** *is*

$$\left[\sum_{j=0}^{m} (-1)^j C_m^j k^{k^{m-j}}\right]^{k^{n-m}}. \tag{8}$$

Of particular interest are the functions $f \in P_n^k$, for which every subfunction $g \in G$ takes as many different values as the function f itself.

Theorem 2. *If $\emptyset \neq M \subset X_f$, $|M| = m$, then the number of functions $f \in P_n^k$ for which* $\textbf{\textit{Rng}}(f) = \textbf{\textit{Rng}}(M, f; g) = q$, $q \leq k$, *for all* g, $g \in G$ *is*

$$C_k^q \left[\frac{\mu_m^k(q)}{C_k^q}\right]^{k^{n-m}} = C_k^q \left[\sum_{s=1}^{q}(-1)^{q-s} C_q^s s^{k^m}\right]^{k^{n-m}}. \tag{9}$$

Proof. From $\mathbf{Rng}(f) = \mathbf{Rng}(M, f; g) = q$ it follows that each subfunction $g \in G$ and the function f assume the same q values.

If we choose q, $q \leq k$, values that can be assumed by each function from P_n^k, then from (1) it follows that the number of functions from P_n^k which assume the chosen q values (property $\mathbf{Q}$) is

$$\sum_{\substack{r_1+r_2+\ldots+r_q=k^n \\ r_i \geq 1,\ i=1,2,\ldots,q}} \frac{k^n!}{r_1! r_2! \ldots r_q!} = \frac{\mu_n^k(q)}{C_k^q} = \mathbf{Q}_{n,k}.$$

The proof is obtained by applying Theorem 1, (4) to the number of functions from P_n^k for which each subfunction $g \in G$ assumes the chosen q values (property $\mathbf{Q}$) and taking into consideration that q values can be chosen among k values in C_k^q ways.

Theorem 3. *If $\emptyset \neq M \subset X_f$, $|M| = m$ then the number of functions $f \in P_n^k$ for which* $\textbf{\textit{Rng}}(f) = \textbf{\textit{Rng}}(M, f) = q$, $1 \leq q \leq k$ *is*

$$C_k^q \left\{ \left[\sum_{i=1}^{q} \frac{C_q^i}{C_k^i} \mu_m^k(i)\right]^{k^{n-m}} - \left[\sum_{i=0}^{q-1} \frac{C_q^i}{C_k^i} \mu_m^k(i)\right]^{k^{n-m}} \right\}. \tag{10}$$

Proof. Number of functions $f \in P_n^k$, that assume at most q ($q \leq k$) fixed values (property $\mathbf{Q}$) is $\displaystyle\sum_{i=1}^{q} \frac{C_q^i}{C_k^i} \mu_n^k(i) = \mathbf{Q}_{n,k}$. It follows from Theorem 1, (4) that the number of functions from P_n^k for which every subfunction $g \in G$ assumes at most q fixed values is $\alpha = \left[\displaystyle\sum_{i=1}^{q} \frac{C_q^i}{C_k^i} \mu_m^k(i)\right]^{k^{n-m}}$.

Obviously $\beta = \left[\displaystyle\sum_{i=1}^{q-1} \frac{C_q^i}{C_k^i} \mu_m^k(i)\right]^{k^{n-m}}$ is the number of functions from P_n^k for which every subfunction $g \in G$ assumes at most $q-1$ among the fixed q, $q > 1$ values, or $\beta = \left[\displaystyle\sum_{i=0}^{q-1} \frac{C_q^i}{C_k^i} \mu_m^k(i)\right]^{k^{n-m}}$ when $q \geq 1$.

Taking into consideration that q values among k values can be chosen in C_k^q ways and that $\alpha - \beta$ is the number of functions from P_n^k, for which at least one subfunction $g \in G$ assumes all q fixed values, i.e. the functions for which $\mathbf{Rng}(f) = \mathbf{Rng}(M, f) = q$, we get the proof of Theorem 3.

Let $M \neq \emptyset$ and $R \neq \emptyset$ be two sets of variables for the function $f \in P_n^k$, where $M \not\subset R$ and H be the set of all subfunctions of f with respect to R, i.e. $H = \{h : h \overset{R}{\prec} f\}$.

Definition 9. ([2]) *The* ***range*** *of M with respect to R of the function f with a subfunction h is the* ***range*** *of the set $M \setminus R$ with respect to the function h.*

The **range** of the set M with respect to R of the function f with a subfunction h will be denoted by $\mathbf{Rng}(M;R,f;h)$ where $\mathbf{Rng}(M;R,f;h)=\mathbf{Rng}(M\backslash R,h)$.

Definition 10. ([2]) *The set $\boldsymbol{Spr}(M;R,f)=\cup_{h\in H}\{\boldsymbol{Rng}(M;R,f;h)\}$ is spectrum of the set M with respect to R for f.*

Obviously $\mathbf{Spr}(M;R,f)\subseteq\{1,2,\ldots,k\}$.

Definition 11. *Let $\boldsymbol{C\text{-}Spr}(M;R,f)$ be the set $\{1^{p_1},2^{p_2},\ldots,k^{p_k}\}$, where p_t, $t=1,\ldots,k$ is the number of all the different tuples of constants for the variables of R, for which from f subfunctions can be obtained, with respect to which the set of variables $M\backslash R$ has range equal to t.*

It is obvious that $p_1+p_2+\ldots+p_k=\lambda_{|R|}=k^{|R|}$, where $p_t\geq 0$, $t=1,\ldots,k$.

Theorem 4. *If $M\subset X_f$, $R\subset X_f$, $|M|=m$, $|R|=r$, $|M\cap R|=s$, then the number of functions $f\in P_n^k$, for which $\boldsymbol{C\text{-}Spr}(M;R,f)=\{1^{p_1},2^{p_2},\ldots,k^{p_k}\}$, is*

$$\frac{k^r!}{p_1!p_2!\ldots p_k!}.\alpha_1^{p_1}.\alpha_2^{p_2}\ldots\alpha_k^{p_k}, \tag{11}$$

where $\alpha_1=k^{k^{n+s-m-r}}$, $\alpha_t=\left[\sum_{i=1}^{t}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}-\left[\sum_{i=1}^{t-1}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}$ *for* $t>1$.

Proof. Associate the number t with each tuple of constants for the variables of R, for which from f subfunctions can be obtained, with respect to which $M\backslash R$ has range equal to t. The number of all the different associations, where with p_1 tuples of constants we associate 1, with p_2 tuples of constants we associate 2, …, with p_k tuples of constants we associate k, is

$$\frac{(p_1+p_2+\ldots+p_k)!}{p_1!p_2!\ldots p_k!}=\frac{k^r!}{p_1!p_2!...p_k!}.$$

To each number t, a subfunction from P_{n-r}^k corresponds with respect to which $M\backslash R$ ($|M\backslash R|=m-s$) has range equal to t. Taking into consideration that these subfunctions can be chosen for $t=1$ ([2], Corollary 2.3) in $\alpha_1=k^{k^{n+s-m-r}}$ different ways, and for $t>1$ ([2], Theorem 2.4) in $\alpha_t=\left[\sum_{i=1}^{t}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}-\left[\sum_{i=1}^{t-1}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}$ ways, we obtain the proof of the theorem.

Theorem 5. *If $M\subset X_f$, $R\subset X_f$, $|M|=m$, $|R|=r$, $|M\cap R|=s$, then the number of functions $f\in P_n^k$, for which $\boldsymbol{Spr}(M;R,f)=\{q_1,q_2,\ldots,q_j\}$, $j\leq k$, is*

$$\sum_{\substack{r_1+r_2+\ldots+r_j=k^r\\ r_t\geq 1,\ t=1,2,\ldots,j}}\frac{k^r!}{r_1!r_2!\ldots r_j!}.\rho_1^{r_1}.\rho_2^{r_2}\ldots\rho_j^{r_j}, \tag{12}$$

where $\rho_t=\left[\sum_{i=1}^{q_t}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}-\left[\sum_{i=0}^{q_t-1}\mu_{m-s}^k(i)\right]^{k^{n+s-m-r}}$, $t=1,\ldots,j$.

Proof. Let us denote by r_t the number of all different tuples of constants for the variables from R, for which from f subfunctions can be obtained, with respect to which the set of variables $M\backslash R$ has range equal to q_t, $t = 1, 2, \ldots, j$.

In this case, for f we have $\mathbf{C\text{-}Spr}(M; R, f) = \{q_1^{r_1}, q_2^{r_2}, \ldots, q_j^{r_j}\}$.

Taking into consideration the fact that each function f for which $\mathbf{C\text{-}Spr}(M; R, f) = \{q_1^{r_1}, q_2^{r_2}, \ldots, q_j^{r_j}\}$, where $r_1+r_2+\ldots+r_j = k^r$, $r_t \geq 1$, $t = 1, 2, \ldots, j$, has $\mathbf{Spr}(M; R, f) = \{q_1, q_2, \ldots, q_j\}$, and (3) and applying Theorem 4, we get the proof of Theorem 5.

When $R = X_f\backslash M$, we have $\mathbf{Spr}(M; R, f)$ $=\mathbf{Spr}(M; X_f\backslash M, f)$ $=\mathbf{Spr}(M, f)$. By applying Theorem 4 (Theorem 5) in this special case, we obtain the number of functions $f \in P_n^k$, for which $\mathbf{C\text{-}Spr}(M, f) = \{1^{p_1}, 2^{p_2}, \ldots, k^{p_k}\}$, ($\mathbf{Spr}(M, f) = \{q_1, q_2, \ldots, q_j\}$, $j \leq k$), where $M \subset X_f$, $|M| = m > 0$, $|R| = r = n - m$, $|M \cap R| = s = 0$.

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Dimiter Stoichkov Kovachev
Department of Computer Sciences
South-West University "Neofit Rilski"
P.O. 79, 2700 Blagoevgrad, Bulgaria
e-mail: dkovach@aix.swu.bg

ВЪРХУ БРОЯ НА НЯКОИ k-ЗНАЧНИ ФУНКЦИИ НА n ПРОМЕНЛИВИ

Димитър Стоичков Ковачев

Нека M и R са множества от аргументи на функцията $f(x_1, x_2, \ldots, x_n) \in P_n^k$, където P_n^k е множеството от всички k-значни функции на n променливи. В тази статия е намерен е броят на функции $f \in P_n^k$, за които

- множеството M е отделимо за f;
- множеството M е **с**-отделимо за f;
- всяка подфункция на f, с аргументи множеството M, приема стойностите на функцията;
- множеството M относно R за функцията f има даден спектър;
- множеството M относно R за функцията f има даден **с**-спектър и др.

Начина на преброяване може да послужи и за „конструиране“ на разгледаните функции.