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REGULAR ONE-PARAMETER SYSTEMS OF TORSES IN EUCLIDEAN SPACE

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In the present paper we consider a class of hypersurfaces of conullity two in Euclidean space. We prove a characterization theorem for the hypersurfaces, which are regular one-parameter systems of torses in terms of thier second fundamental tensor.

1. Preliminaries. For a Riemannian manifold (M^n, g) we denote by T_pM^n the tangent space to M^n at an arbitrary point $p \in M^n$ and by $\mathcal{X}M^n$–the Lie algebra of all C^∞ vector fields on M^n.

Let $(M^n, g, \bar{\Delta})$ be a Riemannian manifold endowed with a two-dimensional distribution $\bar{\Delta}$. Since our considerations are local, we can assume that there is an orthonormal frame field $\{W, \xi\}$ on M^n, which spans $\bar{\Delta}$, i.e. $\bar{\Delta}_p = \operatorname{span}\{W, \xi\}$, $p \in M^n$. We denote by ω and η the one-forms corresponding to W and ξ, respectively:

$$\omega(X) = g(W, X); \quad \eta(X) = g(\xi, X); \quad X \in \mathcal{X}M^n.$$

A Riemannian manifold $(M^n, g, \bar{\Delta})$ with curvature tensor R is said to be *of conullity two* [1], if at every point $p \in M^n$ there exists an orthonormal frame $\{e_1 = W, e_2 = \xi, e_3, \dots, e_n\}$ of the tangent space T_pM^n such that

i) $R(e_1, e_2, e_2, e_1) = -R(e_2, e_1, e_2, e_1) = -R(e_1, e_2, e_1, e_2) = R(e_2, e_1, e_1, e_2) = k(p) \neq 0$;

ii) $R(e_i, e_j, e_k, e_l) = 0$, otherwise.

Let M^n be a hypersurface in Euclidean space E^{n+1}. We denote the standard metric in E^{n+1} by g and its Levi-Civita connection by ∇'. Further, let ∇ be the induced connection on M^n and

$$h(X, Y) = g(AX, Y); \quad X, Y \in \mathcal{X}M^n$$

be the second fundamental tensor of the hypersurface M^n. Hypersurfaces of conullity two are characterized in terms of the second fundamental tensor as follows [4]:

Proposition 1.1. *A hypersurface $(M^n, g, \bar{\Delta})$ in E^{n+1} is of conullity two iff its second fundamental tensor h has the form*

$$h = \lambda\omega \otimes \omega + \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta, \quad \lambda\nu - \mu^2 \neq 0,$$

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where λ, μ and ν are functions on M^n.

A hypersurface M^n, which is a one-parameter system $\{E^{n-1}(s)\}$, $s \in J$, of planes of codimension two, defined in an interval J, is said to be a *ruled hypersurface*. The planes $E^{n-1}(s)$ are called *generators* of M^n. A ruled hypersurface is said to be *developable* (*a torse*), if its normal vector field N is parallel (constant) along each generator. Torses in E^{n+1} are characterized by the following (see [2])

Lemma 1.2. *Let (M^n, g) be a hypersurface in E^{n+1} with second fundamental tensor h. Then M^n is locally a torse iff*

$$h = k\,\omega \otimes \omega,$$

where k and ω are a function and a unit one-form on M^n, respectively.

If N is a unit vector field normal to the torse $T^n = \{E^{n-1}(s)\}$, $s \in J$, then

$$\nabla'_x N = -k\,\omega(X)W; \quad X \in \mathcal{X}T^n,$$

where W is a unit vector field orthogonal to the generators and correspondent to the one-form ω.

Remark 1. Every hyperplane $M^n = E^n$ can be regarded as a torse with $k = 0$. The hyperplanes are trivial torses. We shall only consider non-trivial ones.

Now, let T^{n-1} be a torse in a hyperplane E^n in E^{n+1} $(n \geq 3)$. Such a torse is called *a torse of codimension two*. Hypersurfaces of conullity two, which are one-parameter systems $M^n = \{T^{n-1}(s)\}$, $s \in J$, of torses of codimension two, are characterized in [4]. In general, the second fundamental tensor A of a one-parameter system of torses is not diagonalized. In this paper we study a special class of one-parameter systems of torses, for which the second fundamental tensor A is diagonalized. This is the class of the regular systems of torses. We prove a characterization theorem for hypersurfaces in E^{n+1}, which are regular one-parameter systems of torses.

2. Characterization of hypersurfaces in Euclidean space which are regular one-parameter systems of torses

We consider a torse T^{n-1} of codimension two lying in a hyperplane E^n in E^{n+1} $(n \geq 3)$. The unit vector field orthogonal to the generators of T^{n-1} and its correspondent one-form are denoted by W and ω, respectively. It is clear that the vector field W is determined up to a sign. If l is the unit vector field normal to E^n and n is the unit vector field normal to T^{n-1} in E^n, then the pair $\{l, n\}$ is called *the canonical normal frame* to T^{n-1}. With respect to the canonical normal frame from Lemma 1.2 we have

$$\begin{aligned} &\nabla'_x l = 0; \\ &\nabla'_x n = -k\omega(x)W; \end{aligned} \quad x \in \mathcal{X}T^{n-1}, \tag{2.1}$$

where k is a function on T^{n-1}.

If $\{N, \xi\}$ is an arbitrary normal frame to T^{n-1}, then

$$\begin{aligned} \xi &= \cos\varphi l + \sin\varphi n; \\ N &= -\sin\varphi l + \cos\varphi n, \end{aligned} \tag{2.2}$$

where $\varphi = \measuredangle(n, N)$. The equations (2.1) and (2.2) imply

$$\begin{aligned} &\nabla'_x N = -k\cos\varphi\omega(x)W - d\varphi(x)\xi; \\ &\nabla'_x \xi = -k\sin\varphi\omega(x)W + d\varphi(x)N; \end{aligned} \quad x \in \mathcal{X}T^{n-1}. \tag{2.3}$$

Now, let $M^n = \{T^{n-1}(s)\}$, $s \in J$, be a one-parameter system of torses of codimension two, defined in an interval J. We denote by ξ the vector field orthogonal to the torses $T^{n-1}(s)$ and by N–the unit normal to the hypersurface M^n. Then $\{N, \xi\}$ is a normal frame field to each of the torses $T^{n-1}(s)$ and the equations (2.3) hold good.

We denote by Δ_0 the distribution on M^n, orthogonal to W and ξ, i.e.

$$\Delta_0 = \{x_0 \in \mathcal{X}M^n | \omega(x_0) = \eta(x_0) = 0\}$$

and by Δ the distribution on M^n, orthogonal to ξ, i.e.

$$\Delta = \{x \in \mathcal{X}M^n | \eta(x) = 0\}.$$

Using (2.3), we get

$$\begin{aligned} &\nabla'_{x_0} N = -d\varphi(x_0)\xi, \quad x_0 \in \Delta_0; \\ &\nabla'_W N = -k\cos\varphi W - d\varphi(W)\xi. \end{aligned} \tag{2.4}$$

Definition ([3]). *A one-parameter system $M^n = \{Q^{n-1}(s)\}$, $s \in J$, of surfaces $Q^{n-1}(s)$ of codimension two is said to be regular, if the tangent space T_pQ^{n-1} to an arbitrary surface $Q^{n-1}(s)$ is an eigen space of the second fundamental tensor A of the hypersurface M^n.*

According to the above definition and equalities (2.4) a one-parameter system of torses $M^n = \{T^{n-1}(s)\}$, $s \in J$, is regular iff $d\varphi(x) = 0$, $x \in \Delta$, i.e. φ is a constant on each torse $T^{n-1}(s)$. So, for a regular system of torses we have

$$\begin{aligned} &\nabla'_x N = -k\cos\varphi\omega(x)W; \\ &\nabla'_x \xi = -k\sin\varphi\omega(x)W; \end{aligned} \qquad x \in \Delta. \tag{2.5}$$

If $h(X,Y) = g(AX,Y)$, $X \in \mathcal{X}M^n$, is the second fundamental tensor of the hypersurface M^n, then using the Weingarten formula

$$\nabla'_X N = -A(X), \quad X \in \mathcal{X}M^n,$$

we obtain

$$A(x) = k\cos\varphi\omega(x)W, \quad x \in \Delta.$$

Hence

$$\begin{aligned} &h(x,y) = k\cos\varphi\omega(x)\omega(y) \\ &h(x,\xi) = 0 \end{aligned} \qquad x, y \in \Delta. \tag{2.6}$$

If X and Y are arbitrary vector fields on M^n, then

$$\begin{aligned} &X = x + \eta(X)\xi, \quad x \in \Delta; \\ &Y = y + \eta(Y)\xi, \quad y \in \Delta. \end{aligned} \tag{2.7}$$

So, the equalities (2.6) and (2.7) imply

$$h(X,Y) = \lambda\omega(X) \otimes \omega(Y) + \nu\eta(X) \otimes \eta(Y),$$

where $\lambda = k\cos\varphi$, $\nu = h(\xi,\xi)$.

Thus we proved

Proposition 2.1. *The second fundamental tensor h of a regular one-parameter system of torses has the form*

$$h = \lambda\omega\otimes\omega + \nu\eta\otimes\eta, \quad \lambda\nu \neq 0.$$

Remark 2. In case $\lambda = 0$ or $\nu = 0$ the hypersurface is a torse. This is a trivial case of a regular one-parameter system of torses.

Let (M^n, g, W, ξ) be a hypersurface in E^{n+1} endowed with an orthonormal frame field $\{W, \xi\}$ and second fundamental tensor

$$h = \lambda\omega\otimes\omega + \nu\eta\otimes\eta, \quad \lambda\nu \neq 0. \tag{2.8}$$

We denote by Δ_0 the distribution on M^n, orthogonal to W and ξ. Taking into account (2.8), from the Codazzi equation

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z); \quad X,Y,Z \in \mathcal{X}M^n$$

we obtain the following integrability conditions for hypersurfaces with second fundamental tensor (2.8):

1) $\nabla_{x_0}\xi = \gamma(x_0)W$;

2) $\nabla_{x_0}W = -\gamma(x_0)\xi$;

3) $g(\nabla_W W, x_0) = \dfrac{d\lambda(x_0)}{\lambda}$;

4) $g(\nabla_\xi \xi, x_0) = \dfrac{d\nu(x_0)}{\nu}$;

5) $g(\nabla_W \xi, x_0) = \dfrac{\nu-\lambda}{\nu}\gamma(x_0)$;

6) $g(\nabla_\xi W, x_0) = \dfrac{\nu-\lambda}{\lambda}\gamma(x_0)$;

7) $(\lambda-\nu)g(\nabla_W W, \xi) = d\lambda(\xi)$;

8) $(\lambda-\nu)g(\nabla_\xi \xi, W) = -d\nu(W)$,

where $x_0 \in \Delta_0$ and $\gamma(x_0) = g(\nabla_{x_0}\xi, W)$.

Now we shall prove the main result in the paper

Theorem 2.2. *Let (M^n, g, W, ξ) be a hypersurface in E^{n+1}. Then M^n is locally a regular one-parameter system of torses iff*

i) $h = \lambda\omega\otimes\omega + \nu\eta\otimes\eta$, $\lambda\nu \neq 0$*;*

ii) $\gamma = 0$*;*

iii) $d\left(\dfrac{\operatorname{div}\xi}{\lambda}\right)(x) = 0$*,* $x \in \Delta$*,*

where Δ is the distribution on M^n, orthogonal to ξ.

Proof. I. Let $M^n = \{T^{n-1}(s)\}$, $s \in J$, be a regular one-parameter system of torses and $\{W, \xi\}$ be the orthonormal frame field on M^n such that ξ is orthogonal to the torses $T^{n-1}(s)$ and W is orthogonal to the generators of $T^{n-1}(s)$. According to Proposition 2.1 the second fundamental tensor h of M^n has the form i).

If N is the unit normal to M^n, then the equalities (2.5) hold good. So, for $x_0 \in \Delta_0$ we have $\nabla'_{x_0}\xi = 0$. Hence the one-form γ on Δ_0 is zero.

Further, let $\{e_1, \ldots, e_{n-2}\}$ be an orthonormal basis of Δ_0 at $p \in M^n$. Then from the formula

$$\operatorname{div} \xi = \sum_{i=1}^{n-2} g(\nabla'_{e_i} \xi, e_i) + g(\nabla'_W \xi, W)$$

we find $\operatorname{div} \xi = g(\nabla'_W \xi, W)$. Now the second equality of (2.5) implies

$$\operatorname{div} \xi = -k \sin \varphi.$$

Taking into account that $\lambda = k \cos \varphi$, we find $\tan \varphi = -\dfrac{\operatorname{div} \xi}{\lambda}$. But for a regular system of torses φ is a constant on each torse $T^{n-1}(s)$, so $d(\dfrac{\operatorname{div} \xi}{\lambda})(x) = 0$, $x \in \Delta$.

II. Let M^n be a hypersurface for which the conditions i)–iii) hold good. Taking into account that $\gamma(x_0) = 0$ and the integrability conditions 1) and 5), we get

$$d\eta(x, y) = 0, \quad x, y \in \Delta.$$

Hence the distribution Δ is involutive. So, for every point $p \in M^n$ there exists a unique maximal integral submanifold S_p^{n-1} of Δ containing p. Thus $M^n = \{S^{n-1}(s)\}$, $s \in J$, is a one-parameter system of surfaces $S^{n-1}(s)$ of codimension two. Using i), ii) and the integrability conditions 1) and 5) for M^n, we get

$$\begin{aligned} \nabla'_x N &= -\lambda \omega(x) W, \\ \nabla'_x \xi &= \operatorname{div} \xi \omega(x) W, \end{aligned} \qquad x \in \Delta. \tag{2.9}$$

Denoting $\varphi = \arctan \dfrac{-\operatorname{div} \xi}{\lambda}$ and using iii), we obtain $d\varphi(x) = 0$, $x \in \Delta$, i.e. φ is only a function of the parameter s. So, there exists a function k on M^n, such that

$$\lambda = k \cos \varphi; \quad -\operatorname{div} \xi = k \sin \varphi.$$

Thus the equalities (2.9) take the form

$$\begin{aligned} \nabla'_x N &= -k \cos \varphi \omega(x) W; \\ \nabla'_x \xi &= -k \sin \varphi \omega(x) W; \end{aligned} \qquad x \in \Delta. \tag{2.10}$$

Let $\{l, n\}$ be the frame field of M^n given by

$$\begin{aligned} l &= \cos \varphi \xi - \sin \varphi N; \\ n &= \sin \varphi \xi + \cos \varphi N. \end{aligned} \tag{2.11}$$

Then the equalities (2.10) and (2.11) imply

$$\begin{aligned} \nabla'_x l &= 0; \\ \nabla'_x n &= -k \omega(x) W; \end{aligned} \qquad x \in \mathcal{X} S^{n-1}. \tag{2.12}$$

The first equation of (2.12) shows that there exists a hyperplane E^n with normal l such that S^{n-1} lies in E^n. The second equality of (2.12) and Lemma 1.2 imply that S^{n-1} is locally a torse in E^n with normal n.

Consequently, M^n is locally a regular one-parameter system of torses of codimension two, which are the integral submanifolds of the distribution Δ. $\square$

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РЕГУЛЯРНИ ЕДНОПАРАМЕТРИЧНИ СИСТЕМИ ОТ ТОРСОВЕ В ЕВКЛИДОВО ПРОСТРАНСТВО

Величка Василева Милушева

В настоящата статия изучаваме клас хиперповърхнини в евклидово пространство, за които нулевото пространство на тензора на кривина има коразмерност две. Докозваме характеризационна теорема за хиперповърхнините, които са еднопорометрични системи от торсове, в термините на втората им основна форма.