# ON THE NUMBER OF DOUBLE POINTS IN PLANE $\left(q^{2}+q+2, q+2\right)$-ARCS 

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Let $\mathcal{K}$ be a $\left(q^{2}+q+2, q+2\right)$-arc in $\mathrm{PG}(2, q)$ and let $D$ denote the number of its double points. In this note we improve on the known lower bound for $D$ when $q$ is an odd prime power and demonstrate the nonexistence of some hypothetical $(74,10)$-arcs in $P G(2,8)$. In addition we construct a new (74,10)-arc with 19 double points.

A multiset in the projective geometry $(\mathcal{P}, \mathcal{L})=\operatorname{PG}(r, q)$ is a mapping $\mathcal{K}: \mathcal{P} \rightarrow \mathbb{N}_{0}$ from the points of the geometry into the non-negative integers. This mapping can be extended to the subsets of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q})=\sum_{P \in \mathcal{Q}} \mathcal{K}(P), \mathcal{Q} \subseteq \mathcal{P}$. The integer $\mathcal{K}(\mathcal{Q})$ is called the multiplicity of the set $\mathcal{Q}$. Sets $\mathcal{Q}$ with $\mathcal{K}(\mathcal{Q})=i$ are called $i$-sets. In particular, points, lines and hyperplanes of multiplicity $i$ are called $i$-points, $i$-lines and $i$-hyperplanes, respectively. For a multiset $\mathcal{K}$ in $\operatorname{PG}(r, q)$, let $a_{i}$ denote the number of the $i$-hyperplanes. The sequence $\left\{a_{i}\right\}_{i \geq 0}$ is then called the spectrum of $\mathcal{K}$.

Let $\mathcal{P}$ and $\mathcal{L}$ be the sets of points and lines in $\operatorname{PG}(2, q)$, respectively. The multiset $\mathcal{K}$ is called a $(k, n)$-arc if $\mathcal{K}(\mathcal{P})=k, \mathcal{K}(l) \leq n$ for any line $l \in \mathcal{L}$, and there is a line with $\mathcal{K}(l)=n$. In this paper we are interested in $\left(q^{2}+q+2, q+2\right)$-arcs in the projective plane of order $q$. Arcs with such parameters arise in various classification problems for optimal linear codes of higher dimension [2] since they correspond to linear $\left[q^{2}+q+2,3, q^{2}\right]_{q}$ codes.

In [1] S. Ball et al. initiated the research on $\left(q^{2}+q+2, q+2\right)$-arcs. Although they did not give a complete classification of such arcs, they constructed several infinite families of arcs and proved nonexistence results under some special restrictions.

Throughout the paper $\mathcal{K}$ will denote a $\left(q^{2}+q+2, q+2\right)$-arc in $\operatorname{PG}(2, q)$. It is easily checked, that $\mathcal{K}(P) \leq 2$ for any point $P$. The point sets $\{P \mid \mathcal{K}(P)=i\}, i=0,1,2$ are denoted by $\mathcal{K}_{i}$. We set $R=\left|\mathcal{K}_{0}\right|, S=\left|\mathcal{K}_{1}\right|, D=\left|\mathcal{K}_{2}\right|$. Obviously $D=R+1$, $S=q^{2}+q-2 R$. The following lemma describes the distribution of 0-points and 2-points on any line in $\mathrm{PG}(2, q)$.

Lemma 1 [1]. Let $\mathcal{K}$ be a $\left(q^{2}+q+2, q+2\right)$-arc in $\operatorname{PG}(2, q)$ and let $l$ be a line. Suppose $l$ containes $r 0$-points and $d$ 2-points. Then $r \geq d-1$. Moreover, if $d>0$ then $r=d-1$.

The following identities for the spectrum of a $\left(q^{2}+q+2, q+2\right)$-arc are equivalent to the first three applying the MacWilliams identities for the code corresponding to this arc, or by counting the number of hyperplanes, the number flags consisting of point
and hyperplane and the number flags consisting of a pair of points and a hyperplane, respectively.

$$
\begin{gather*}
\sum_{i=0}^{q+2} a_{i}=q^{2}+q+1  \tag{1}\\
\sum_{i=1}^{q+2} i a_{i}=(q+1)\left(q^{2}+q+2\right)  \tag{2}\\
\sum_{i=2}^{q+2}\binom{i}{2} a_{i}=\binom{q^{2}+q+2}{2}+D q \tag{3}
\end{gather*}
$$

One possible approach to $\left(q^{2}+q+2, q+2\right)$-arcs is to try to find all values of $D$ (or, equivalently, $R$ ) for which such arcs do exist. For instance in case of $q=2$ it is easy to check that $R$ can be 0,1 or 3 [1]. When $q>2$ we have the following upper bound for $R$, consequently for $D$ and $S$ :

Theorem 2 [1]. Let $\mathcal{K}$ be a $\left(q^{2}+q+2, q+2\right)$-arc with $q>2$ and let $f$ denote the number of lines free of 2-points, or free lines for short. Then $f \geq q$. Equality holds exactly when $R=\binom{q}{2}$.

It turns out that such arcs really do exists for each $q$. They are described in the following construction.

The dual arc construction [1]. Take a set of $q$ lines in $\operatorname{PG}(2, q)$ or, equivalently, a $q$-arc in the dual plane. Let the points of intersection of these lines be the 0-points, let the remaining points on the $q$ lines be 1-points, and let everything else be 2-points. The so-defined multiset is a $\left(q^{2}+q+2, q+2\right)$-arc with $D=\binom{q}{2}+1$.

If $D$ is close to but less than $\binom{q}{2}+1,\left(q^{2}+q+2, q+2\right)$-arcs do not exist in general. As a matter of fact, an arc with $D=\binom{q}{2}$ exists iff $q=2$ or $q=4$, i.e. in all other cases $D<\binom{q}{2}$.

In [1] it is proved that for no $\left(q^{2}+q+2, q+2\right)$-arc the parameter $R$ satisfies the inequalities:

$$
\begin{equation*}
\frac{(q+1)\left(\sqrt{8 q^{2}+1}-2 q-1\right)}{2}<R \leq \frac{q(q-1)}{2}-1 \tag{4}
\end{equation*}
$$

This gives approximately

$$
\begin{equation*}
(\sqrt{2}-1) q^{2} \lesssim R \lesssim \frac{q^{2}}{2} \tag{5}
\end{equation*}
$$

The main result in this paper is our Theorem 4, which improves on the lower bounds in (4) and (5). Our proof relies on the following result from [1], which shows that when the number of double points is big enough then $\mathcal{K}$ is a divisible arc (i.e. $\mathcal{K}(l)$ is the same modulo some power of $p$ for all lines $l$ ).

Theorem 3 [1]. Let $\mathcal{K}$ be a $\left(q^{2}+q+2, q+2\right)$-arc in $\mathrm{PG}(2, q), q=p^{m}$ and suppose that $(q-1) p^{t-1}<D$. Then for any line $l, \mathcal{K}(l) \equiv 2\left(\bmod p^{t}\right)$.

Theorem 4. There are no $\left(q^{2}+q+2, q+2\right)$-arcs in $\mathrm{PG}(2, q), q=p^{m}$, with

$$
\begin{equation*}
\frac{q(q-1)}{p}<D \leq \frac{q(q-1)}{2} . \tag{6}
\end{equation*}
$$

Proof. Let $\mathcal{K}$ be a $\left(q^{2}+q+2, q+2\right)$-arc with $p^{m-1}\left(p^{m}-1\right)<D \leq\binom{ q}{2}$ and let $l$ be a line in $\operatorname{PG}(2, q)$. By Theorem 3 we have the congruence $\mathcal{K}(l) \equiv 2\left(\bmod p^{m}\right)$. Hence $\mathcal{K}(l)=2$ or $\mathcal{K}(l)=p^{m}+2$. By (1) and (2), we get that the spectrum of $\mathcal{K}$ is $a_{2}=q$, $a_{q+2}=q^{2}+1, a_{i}=0$ for all $i \neq 2, q+2$. Any two 2 -lines meet in a 0 -point, for if we assume they meet in a 1 -point $P$, we get a contradiction by counting the multiplicities of the lines through $P$. In a similar way we can rule out the possibility of three 2 -lines being concurrent.

Therefore the number of 0-points is $R=\frac{1}{2} a_{2}(q-1)=\binom{q}{2}$, which implies $D=$ $\binom{q}{2}+1$, a contradiction to our initial assumption.

Corollary 5. If $\mathcal{K}$ is a $\left(q^{2}+q+2, q+2\right)$-arc with $D \leq\binom{ q}{2}$, then $D \leq \frac{q(q-1)}{p}$.
The inequality of the above corollary gives asymptotically

$$
\begin{equation*}
D \lesssim \frac{q^{2}}{p} \tag{7}
\end{equation*}
$$

which is better than (4) for all $p \geq 3$. This bound represents a major improvemant for larger values of $p$.

It is clear that Theorem 4 does not work if the characteristic of the field is $p=2$. That is why we focus our attention on the first field of characteristic 2 for which the problem of the characterization of $\left(q^{2}+q+2, q+2\right)$-arcs is not solved. Thus we are looking for $(74,10)$-arcs in $\operatorname{PG}(2,8)$.

Let $\mathcal{K}$ be a $(74,10)$-arc in $\operatorname{PG}(2,8)$ with $D<29$. By (4), the number of double points $D \leq 26$. Assume that $D>14$. Then for any line $l$, we get the congruence $\mathcal{K}(l) \equiv 2$ $(\bmod 4)$ (Theorem 3), i.e. $\mathcal{K}(l)=2,6$ or 10 . For the spectrum of the arc from the identities (1), (2) and (3) we get $2 a_{2}+a_{6}=16$ and $2 a_{2}=D-13$. As the number $f$ of free lines is greater or equal to 11 , it follows that $a_{2} \leq 5$ and $D \leq 23(R \leq 22)$.

All the possible values, in the case under consideration, of the parameters $a_{2}, a_{6}, a_{10}$, $R, S$ and $D$ are summarizeded in the following table:

|  | $a_{2}$ | $a_{6}$ | $a_{10}$ | $R$ | $S$ | $D$ |
| :---: | ---: | ---: | ---: | ---: | ---: | :---: |
| $(a)$ | 1 | 14 | 58 | 14 | 44 | 15 |
| $(b)$ | 2 | 12 | 59 | 16 | 40 | 17 |
| $(c)$ | 3 | 10 | 60 | 18 | 36 | 19 |
| $(d)$ | 4 | 8 | 61 | 20 | 32 | 21 |
| $(e)$ | 5 | 6 | 62 | 22 | 28 | 23 |

Cases $(d)$ and ( $e$ ) are easily ruled out by counting the number of 0 -points contained in a the 2 -lines. To this end, note the any two 2 -lines meet in a 0 -point and no three 2 -lines are concurrent. If this were the case we would get a contradiction by counting the multiplicities of the lines through the point of concurrence. Hence the 2-lines contain at least $7 a_{2}-\binom{a_{2}}{2}$ points. This gives 210 -points in case $(d)$ and 250 -points in case (e), a contradiction in both cases.

In case $(c)$ we can construct a $(74,10)$-arc. Let us start with a maximal $(28,4)$-arc $\mathcal{M}$ in $\operatorname{PG}(2,8)$. Such an arc can be constructed by taking the points which are not incident with ten lines forming a hyperoval in the dual plane. It is easily checked that all lines are either meet the $(28,4)$-arc in 4 points or are external to the arc.

Let us choose three non-concurrent 4 -lines (with respect to $\mathcal{M}$ ), $l_{1}, l_{2}$ and $l_{3}$ say. Now we define a new arc $\mathcal{K}$ in the following way. Let all points of $\mathcal{M}$ not on $l_{1}, l_{2}$ or $l_{3}$ be 2-points; let the points on $l_{1}, l_{2}, l_{3}$ that are not in $\mathcal{M}$ and the three points of intersection $l_{i} \cap l_{j}$ be 0 -points and let all the remaining points be 1-points.

It is now easily checked that $\mathcal{K}$ is a $(74,10)$-arc. The three lines $l_{1}, l_{2}, l_{3}$ are 2 -lines. The lines through the points $l_{i} \cap l_{j}$ either meet the third special line in a point from $\mathcal{M}$ and contain thus exactly one 0 -point and two 2 -points, or else contain exactly two 0 - and three 2 -points. In both cases we get a 10 -line. Finally, we consider the lines meeting the $l_{i}$ 's in points different from $l_{i} \cap l_{j}$. Let $l$ be such a line and let $l$ have a non-empty intersection with $\mathcal{M}$. The line $l$ meets the $l_{i}$ 's in 3,2 , 1 , or 0 points of $\mathcal{M}$. By our construction in these cases we have $3,2,1$ or 00 -points, and $4,3,2$ or 12 -points, respectively. Hence $l$ is a 10 -line. If $l$ is external to $\mathcal{M}$ it meets the $l_{i}$ 's in 0 -points and is thus a 6 -line.

## REFERENCES

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## ВЪРХУ БРОЯ НА ДВОЙНИТЕ ТОЧКИ В РАВНИННИ $\left(q^{2}+q+2, q+2\right)$-АРКИ <br> Ася Петрова Русева

Нека $\mathcal{K}$ е $\left(q^{2}+q+2, q+2\right)$-арка в $\operatorname{PG}(2, q)$ и нека означим с $D$ броя на двойните и́ точки. В тази статия подобряваме известната долна граница за $D$ в случая на полета с нечетна характеристика и доказваме несъществуването на някои хипотетични $(74,10)$-арки в $\mathrm{PG}(2,8)$. В статията е описана и конструкция на нова $(74,10)$-арка с 19 двойни точки.

