# CONVEX FUNCTIONS AND PARETO OPTIMALITY CRITERION 

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#### Abstract

In the presented paper the Pareto optimal distributions are analyzed. The basic concept of welfare economy is examined here - Pareto optimality criterion. We consider a mathematical model of exchange economy. The presented theorems do not use the prices of the goods and budgetary limitations of the agents. The theorems are based only on the utility of the goods and the status quo of the agents.


1. Introduction. Welfare economy is a theory where consumption and production are examined on one and same scale. The simultaneous existence of a number of consumers and producers equal in rights is taking place under the condition of equilibrium with optimality distribution of goods of both for consumers and producers. The problems of equilibrium and optimality are basic for welfare economy.

A basic characteristic of the Equilibrium State is given by the Pareto optimality criterion. A distribution of goods is Pareto optimal if and only if in the process of transition into another distribution there is no deterioration of the status quo of the agents. Measure for the status quo of the agent will be his or her utility function. Vilfredo Pareto (1878-1923) lays out the criterion: if in the process of distribution of goods between the agents in an exchange economic system the welfare of one single agent increases, without decreasing the welfare of all the other agents, then the welfare of the system as a whole increases. We have the definition: the distribution of goods is Patero optimal if and only if it is not possible for the welfare of a certain agent to be improved without involving the worsening of the welfare of another agent.

It is proved that the equilibrium distributions of goods are Pareto optimal. We will examine a number of characteristics of the Pareto optimality distributions without using the fact of equilibrium. This is an important issue because Pareto optimality distributions do not use the prices of the goods and budgetary limitations of the agents.
2. Formulation of the problem. We are examining a mathematical model of exchange economy with $n \geq 2$ agents and $m \geq 2$ goods. The agents exchange between each other goods. Let $A$ be a set of agents, let $G$ be a set of goods, let $L=n . m$ and let each agent own initial property demonstrated by the vector $v^{i}\left(v_{1}^{i}, \ldots, v_{j}^{i}, \ldots, v_{m}^{i}\right) \in \Re_{+}^{m}$, where the number $v_{j}^{i} \geq 0$ is the quantity of $g_{j} \in G$ property of $a_{i} \in A$. The vector $\Omega=\sum_{i=1}^{n} v^{i}$ we will call the vector of common goods, where $\Omega\left(\Omega_{1}, \Omega_{2}, \ldots, \Omega_{m}\right) \in \Re_{+}^{m}$.

Definition 1. The set $D=\left\{X\left(x^{1}, x^{2}, \ldots, x^{n}\right) \in \Re_{+}^{L}: \sum_{i=1}^{n} x^{i}=\Omega\right\}$ we will call the set of distributions, where $a_{i} \in A$ is owner of $x^{i} \in \Re_{+}^{m}$.

The initial property of the agents is given by the vector $V\left(v^{1}, v^{2}, \ldots, v^{n}\right) \in D$.
Theorem 1. The set $D$ is nonempty, convex and compact set in $\Re_{+}^{L}$.
From Definition 1 it follows the proof of Theorem 1.
If $X \in D$ and $a_{i} \in A$, then let us denote $P_{i}(X)=x^{i}\left(x_{1}^{i}, x_{2}^{i}, \ldots, x_{m}^{i}\right)$ and let each agent $a_{i} \in A$ has an utility function $u_{i}: D \rightarrow \Re_{+}$with the following characteristics:

Assumption 1. The function $u_{i}$ is continuous in $D$;
Assumption 2. If $X, Y \in D$ and $P_{i}(X)=P_{i}(Y)$, then $u_{i}(X)=u_{i}(Y)$;
Assumption 3. If $X, Y \in D, P_{i}(X) \neq P_{i}(Y)$ and $P_{i}(X) \geq P_{i}(Y)$, then $u_{i}(X)>u_{i}(Y)$;
Assumption 4. If $X, Y \in D, P_{i}(X) \neq P_{i}(Y), u_{i}(Y)>u_{i}(X)$ and $\alpha \in(0 ; 1)$, then $u_{i}(\alpha X+(1-\alpha) Y)>u_{i}(X)$.

Definition 2. The function $U: D \rightarrow \Re_{+}^{n}$ we will call collective utility function if and only if $U(X)=\left(u_{1}(X), u_{2}(X), \ldots, u_{n}(X)\right)$ for $X \in D$, where the functions $u_{i}$ for $i=1, \ldots, n$ are the utility functions of the agents.

Theorem 2. The function $U$ is continuous in $D$.
Proof. From the continuity of the functions $u_{i}$ for $i=1, \ldots, n$ in $D$ it follows that the function $U$ is continuous in $D$.

Definition 3. We will call the distribution $X \in D$ Pareto optimal if and only if $\exists Y \in D$ such that $\forall a_{i} \in A, u_{i}(Y) \geq u_{i}(X)$ and $\exists a_{k} \in A, u_{k}(Y)>u_{k}(X)$. The set of Pareto optimal distributions of $D$ will be denoted by $P$.

Therefore we have $X \in P$ if and only if $\left\{Y \in D: \forall a_{i} \in A, u_{i}(Y) \geq u_{i}(X)\right.$ and $\left.\exists a_{k} \in A, u_{k}(Y)>u_{k}(X)\right\}=\emptyset$.

It becomes clear from the definition that the Pareto optimality is not related to the prices of goods, but is defined only by the utility functions of the agents.

## 3. Existence of Pareto optimality.

Definition 4. Let $C=\left\{\left\{c_{i} \geq 0: i=1, \ldots, n\right\} \subset \Re: \sum_{i=1}^{n} c_{i}=1\right\}$. We will call

$$
D_{c}=\left\{X \in D: \sum_{i=1}^{n} c_{i} u_{i}(X) \geq \sum_{i=1}^{n} c_{i} u_{i}(Y) \forall Y \in D, c \in C\right\}
$$

the set of maximal distributions of $D$.
Theorem 3. The set $D_{c}$ is nonempty and compact subset of $D$.
Proof. Let function $f: \rightarrow \Re_{+}$is defined by $f(X)=\sum_{i=1}^{n} c_{i} u_{i}(X)$ for $X \in D$. From the continuity of the functions $u_{i}$ for $i=1, \ldots, n$ it follows that the function $f$ is continuous in $D$. The set $D$ is compact, therefore $\exists X \in D$ such that $f(X)=\sup \{f(Y): Y \in D\}$. We have $D_{c} \neq \emptyset$. From the continuity of the function $f$ it follows that the set $D_{c}$ is a closed subset of the compact set $D$, therefore $D_{c}$ is a compact subset of $D$. The theorem is proved.

Theorem 4. $D_{c} \subset P$.
Proof. Let $X \in D_{c}$ and let us assume that $X \notin P$. Then $\exists Y \in D$ such that $\forall a_{i} \in A$ $u_{i}(Y) \geq u_{i}(X)$ and $\exists a_{k} \in A u_{k}(Y)>u_{k}(X)$ and let $f(X)=\sum_{i=1}^{n} c_{i} u_{i}(X)$, therefore $f(Y) \geq f(X)$.

If $c_{k}>0$, then $f(Y)>f(X)$, which is in contradiction with $X \in D_{c}$.
If $c_{k}=0$, then $f(Y)=\sum_{i=1}^{n} c_{i} u_{i}(Y)=\sum_{i=1, i \neq k}^{n} c_{i} u_{i}(Y)$. From $u_{k}(Y)>u_{k}(X) \geq 0$ it follows that $P_{k}(Y) \neq 0$. From $\sum_{i=1}^{n} c_{i}=1$ it follows that $\exists a_{j} \in A$ such that $c_{j}>0$. From $P_{k}(Y) \neq 0$ it follows that $P_{j}(Y) \neq \Omega$. Therefore $\exists Z \in D$ such that $P_{j}(Z)>P_{j}(Y)$, $P_{k}(Z)<P_{k}(Y)$ and $P_{i}(Z)=P_{i}(Y)$ for $i \neq j$ and $i \neq k$. Finally, we have $f(Z)>f(Y) \geq$ $f(X)$, which is in contradiction with $X \in D_{c}$.

We have in result $D_{c} \subset P$. The theorem is proved.
From the above two theorems it follows that the Pareto optimal distributions exist in the economic system and $\bigcup_{c \in C} D_{c} \subset P$.

## 4. Convex functions and Pareto optimality

Definition 5. If $X \in D$ and $a_{i} \in A$, then the set $R_{i}(X)=\left\{Y \in D: u_{i}(Y) \geq u_{i}(X)\right\}$ we will call set of preferences of $a_{i} \in A$.

It is clear that the set $R_{i}(X)$ is compact and $X \in R_{i}(X) \forall a_{i} \in A$, therefore $\bigcap_{i=1}^{n} R_{i}(X)$ is compact and $X \in \bigcap_{i=1}^{n} R_{i}(X)$.

Theorem 5. Let $X \in D, X \in P$ if and only if $\{X\}=\bigcap_{i=1}^{n} R_{i}(X)$.
Proof. Let $X \in P$. We have $X \in R_{i}(X) \forall a_{i} \in A$, therefore $\{X\} \subset \bigcap_{i=1}^{n} R_{i}(X)$. Let $Y \in \bigcap_{i=1}^{n} R_{i}(X), \alpha \in(0,1)$ and $Z=\alpha X+(1-\alpha) Y$ and let us assume that $X \neq Y$.

If $a_{i} \in A$ and $P_{i}(X)=P_{i}(Y)$, then $P_{i}(Z)=P_{i}(X)$. Therefore $u_{i}(Z)=u_{i}(X)$.
If $a_{i} \in A$ and $P_{i}(X) \neq P_{i}(Y)$, then $u_{i}(Z)>\min \left(u_{i}(X), u_{i}(Y)\right)=u_{i}(X)$.
From $X \neq Y$ it follows that $\exists a_{k} \in A$ such that $P_{k}(X) \neq P_{k}(Y)$. In result, $\forall a_{i} \in A$ $u_{i}(Z) \geq u_{i}(X)$ and $u_{k}(Z)>u_{k}(X)$, which contradicts the condition $X \in P$. Therefore $X=Y$, i.e. $\bigcap_{i=1}^{n} R_{i}(X) \subset\{X\}$. Finally, we have $\{X\}=\bigcap_{i=1}^{n} R_{i}(X)$.

Let $\{X\}=\bigcap_{i=1}^{n} R_{i}(X)$. Let us assume that $X \notin P$, therefore $\exists Y \in D$ such that $\forall a_{i} \in A \quad u_{i}(Y) \geq u_{i}(X)$ and $\exists a_{k} \in A u_{k}(Y)>u_{k}(X)$. In result $Y \in \bigcap_{i=1}^{n} R_{i}(X)=\{X\}$, therefore $X=Y$. This contradicts $u_{k}(Y)>u_{k}(X)$, therefore $X \in P$. The theorem is proved.

Definition 6. The utility function $u_{i}$ of $a_{i} \in A$ is convex if and only if $\forall X, Y \in D$, $P_{i}(X) \neq P_{i}(Y)$ and $\forall \alpha \in[0 ; 1], u_{i}(\alpha X+(1-\alpha) Y) \geq \alpha u_{i}(X)+(1-\alpha) u_{i}(Y)$.

Theorem 6. If the utility functions of the agents $u_{i}$ for $i=1 \ldots, n$ are convex, then $\bigcup_{c \in C} D_{c}=P$.

Proof. Let $X \in P$ and $S=\left\{U(X)+s: s \in \Re_{+}^{n} \backslash\{0\}\right\}$. It is clear that $S \neq \emptyset$.
We will prove that the set $S$ is convex. Let $S_{1}, S_{2} \in S$ and $\alpha \in[0,1]$, then $\exists s_{1}, s_{2} \in$ $\Re_{+}^{n} \backslash\{0\}$ such that $S_{1}=U(X)+s_{1}$ and $S_{2}=U(X)+s_{2}$. For this convex combnination we have $\alpha S_{1}+(1-\alpha) S_{2}=U(X)+\left(\alpha s_{1}+(1-\alpha) s_{2}\right)$, therefore $\alpha S_{1}+(1-\alpha) S_{2} \in S$, i.e. the set $S$ is convex.

Let $B=\left\{U(Y)-b: Y \in D\right.$ and $\left.b \in \Re_{+}^{n} \backslash\{0\}\right\}$. It is clear that $B \neq \emptyset$.
We will prove that the set $B$ is convex. Let $\mathrm{B}_{1}, B_{2} \in B$ and $\alpha \in[0 ; 1]$, then $\exists Y_{1}, Y_{2} \in D$ and $\exists b_{1}, b_{2} \in \Re_{+}^{n} \backslash\{0\}$ such that $B_{1}=U\left(Y_{1}\right)-b_{1}$ and $B_{2}=U\left(Y_{2}\right)-b_{2}$. For its convex combnination we have $\alpha B_{1}+(1-\alpha) B_{2}=\left(\alpha U\left(Y_{1}\right)+(1-\alpha) U\left(Y_{2}\right)\right)-\left(\alpha b_{1}+(1-\alpha) b_{2}\right)$. Let $Y=\alpha Y_{1}+(1-\alpha) Y_{2}$. From the convexity of the set $D$ it follows that $Y \in D$.

If $a_{i} \in A$ and $P_{i}\left(Y_{1}\right)=P_{i}\left(Y_{2}\right)$, then $u_{i}(Y)=u_{i}\left(Y_{1}\right)=u_{i}\left(Y_{2}\right)$.
If $a_{i} \in A$ and $P_{i}\left(Y_{1}\right) \neq P_{i}\left(Y_{2}\right)$, then from the convexity of the utility function $u_{i}$ it follows that $u_{i}(Y) \geq \alpha u_{i}\left(Y_{1}\right)+(1-\alpha) u_{i}\left(Y_{2}\right)$.

Finally, we have $U(Y) \geq \alpha U\left(Y_{1}\right)+(1-\alpha) U\left(Y_{2}\right)$.
Let $b=U(Y)-\left(\alpha U\left(Y_{1}\right)+(1-\alpha) U\left(Y_{2}\right)\right)$, therefore $b \in \Re_{+}^{n} \backslash\{0\}$. In result we have $\alpha B_{1}+(1-\alpha) B_{2}=U(Y)-\left(\alpha b_{1}+(1-\alpha) b_{2}+b\right)$, therefore $\alpha B_{1}+(1-\alpha) B_{2} \in B$, i.e. the set $B$ is convex.

We will prove that $S \cap B=\emptyset$. Let us assume that $S \cap B \neq \emptyset$, therefore $\exists Y \in D$ and $\exists s, b \in \Re_{+}^{n} \backslash\{0\}$ such that $U(X)+s=U(Y)-b$. From $s, b \neq 0$ it follows that $\forall a_{i} \in A$ $u_{i}(Y) \geq u_{i}(X)$ and $\exists a_{k} \in A \quad u_{k}(Y)>u_{k}(X)$, which contradicts the condition $X \in P$, therefore $S \cap B=\emptyset$.

From the theorem for detachability of sets it follows that $\exists\left\{q_{i}: i=1, \ldots, n\right\} \subset \Re$ such that $\sum_{i=1}^{n} q_{i}^{2} \neq 0$ and $\exists c \in \Re$ such that $\forall s\left(s_{1}, s_{2}, \ldots, s_{n}\right), b\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \Re_{+}^{n} \backslash\{0\}$ we have

$$
\sum_{i=1}^{n} q_{i}\left(u_{i}(X)+s_{i}\right) \geq c \geq \sum_{i=1}^{n} q_{i}\left(u_{i}(Y)-b_{i}\right) \text { for all } Y \in D
$$

Therefore $\sum_{i=1}^{n} q_{i} u_{i} \geq c \geq \sum_{i=1}^{n} q_{i} u_{i}(Y)$ for all $Y \in D$ and we have in result an equality at $X=Y$.

We will prove that $q_{i} \geq 0 \forall i \in[1, n]$. The inequality $\sum_{i=1}^{n} q_{i}\left(u_{i}(X)+s_{i}\right) \geq c$ holds for $s\left(s_{1}, s_{2}, \ldots, s_{n}\right) \in \Re_{+}^{n} \backslash\{0\}$. Let us assume that $q_{k}<0$ for $k \in[1, n]$. The inequality $\sum_{i=1}^{n} q_{i}\left(u_{i}(X)+s_{i}\right) \geq c$ holds for

$$
s_{k}>\frac{c-\sum_{i=1, j \neq k}^{n} q_{i}\left(u_{i}(X)+s_{i}\right)}{q_{k}}-u_{k}(X) .
$$

In result we have

$$
\begin{gathered}
u_{k}(X)+s_{k}>\frac{c-\sum_{i=1, j \neq k}^{n} q_{i}\left(u_{i}(X)+s_{i}\right)}{q_{k}}-u_{k}(X) ; \\
q_{k}\left(u_{k}(X)+s_{k}\right)<c-\sum_{i=1, j \neq k}^{n} q_{i}\left(u_{i}(X)+s_{i}\right)
\end{gathered}
$$

$$
\sum_{i=1, j \neq k}^{n} q_{i}\left(u_{i}(X)+s_{i}\right)<c
$$

This leads to a contradiction, therefore $q_{i} \geq 0 \forall i \in[1, n]$. From $\sum_{i=1}^{n} q_{i}^{2} \neq 0$ it follows that $\sum_{i=1}^{n} q_{i}>0$. Let $c_{i}=\frac{q_{i}}{\sum_{i=1}^{n} q_{i}} \geq 0$ for $i=1, \ldots, n$, therefore $\sum_{i=1}^{n} c_{i}=1$ and $\sum_{i=1}^{n} c_{i} u_{i}(X) \geq$ $\sum_{i=1}^{n} c_{i} u_{i}(Y)$ for all $Y \in D$.

Finally, we have $\mathrm{X} \in \bigcup_{c \in C} D_{c}$. From theorem 4 it follows that $\bigcup_{c \in C} D_{c}=P$. The theorem is proved.

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## ИЗПЪКНАЛИ ФУНКЦИИ И КРИТЕРИЯ ЗА ОПТИМАЛНОСТ ПО ПАРЕТО

## Здравко Димитров Славов

В представената статия се анализират разпределенията оптимални по Парето. Предложена е една от основните концепции на икономика на благосъстоянието - критерия за оптималност по Парето. Разгледан е математически модел на разменна икономика. Изложените теореми не използват цени на благата и бюджетни ограничения на агентите. Теоремите се основават само на полезността на благата и предпочитанията на агентите.

