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# VECTOR SLATER SADDLE POINTS AND MAXIMINS FOR SOME TYPES MULTICRITERIAL ANTAGONISTIC GAMES

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In the present paper, some results about the multicriterial antagonistic games with separable pay-off function [1, p. 81-91] have been generalized in Lemma 2 and Theorem 1. Theorem 2 finds a more wide class of games, including the separable ones, for which Lemma 2 and Theorem 1 are valid. As a corollary here, the results of [1] have been obtained. Example 1 shows that Theorem 2 cannot be generalized for arbitrary vectors with nonnegative components. Theorem 1 cannot be generalized for the class of the quadratic multicriterial antagonistic games, considered in Example 2, even in the case when $\varepsilon$ is zero vector. A similar quadratic game, which does not satisfy the conditions of Lemma 2 and Theorem 1 has been considered in Example 3, but the assertions of Theorem 1 are true for this game.

**Introduction.** In the present paper, some properties of some types multicriterial antagonistic games have been considered. The different optimal solutions of the multicriterial control problems and their properties have been studied in detail in [5]. More of the results, presented in [5], are transferred to the multicriterial antagonistic games. For this purpose, the corresponding solutions of a multicriterial antagonistic game are defined. As a solution, there can be taken both the vector Slater, Pareto, etc. saddle ($\varepsilon$-saddle) point and the strategy, corresponding to the vector (Slater, Pareto, etc.) maximin (minimax) or $\varepsilon$-maximin ($\varepsilon$-minimax) solution. All these solutions have been defined and well studied and described, for example in [1]. A lot of these results are applied for differential games in [3,2,4].

**1. Basic definitions and assumptions.** We consider an antagonistic multicriterial game with a pay-off function

$$f(x,y) = (f_1(x,y),\ldots,f_N(x,y)), \qquad x \in X,\ y \in Y. \tag{1}$$

The first player, choosing the strategy $x \in X$, strives to maximize all the components of the vector pay-off function (1) and the second player, choosing $y \in Y$, strives to minimize all these components. We suppose also, that $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ are bounded sets and the functions $f_j(x,y)$ are defined, bounded and continuous for all $(x,y) \in X \times Y$.

We suppose that $\varepsilon \in \mathbb{R}^N_{\geqq}$, i.e. the arbitrary fixed vector of $\mathbb{R}^N$ has nonnegative components.

**Definition 1.** *The point $(x_s^\varepsilon, y_s^\varepsilon)$ is called an $\varepsilon$-Slater saddle point for the game*

$$\begin{aligned}&\Gamma = \langle X, Y, f(x,y)\rangle, \text{ if}\\ &f(x,y_s^\varepsilon) - \varepsilon \not> f(x_s^\varepsilon, y_s^\varepsilon) \not> f(x_s^\varepsilon, y) + \varepsilon \ \forall\ x \in X \text{ and } \forall\ y \in Y.\end{aligned} \tag{2}$$

The set of all vector $\varepsilon$-Slater saddle points is denoted by $S^{\varepsilon}$.

**Definition 2.** *Let $x \in X$ $(y \in Y)$ be an arbitrary point. Then the set*

$$Y_s^{\varepsilon}(x) = \{y_* \in Y \big| f(x, y_*) \not> f(x, y) + \varepsilon, \ \forall y \in Y\}$$

$(X_s^{\varepsilon}(y) = \{x^* \in X \big| f(x^*, y) \not< f(x, y) - \varepsilon, \ \forall x \in X\})$ *represents the set of all $\varepsilon$-Slater minimal (maximal) strategies with relation to $f$,* [1, p. 11].

**Definition 3.** $\hat{Y}_s^{\varepsilon} = \{\hat{x}^* \in X \big| f(\hat{x}^*, \hat{y}_s^{\varepsilon}(\hat{x}^*)) \not< f(x, y_s^{\varepsilon}(x)) - \varepsilon, \ \forall x \in X, \ y_s^{\varepsilon}(x) \in Y_s^{\varepsilon}(x), \ \hat{y}_s^{\varepsilon}(\hat{x}^*) \in Y_s^{\varepsilon}(\hat{x}^*)\}$ *is the set of all vector $\varepsilon$-Slater maximin solutions (strategies) for game* (2)*. By analogy, the set of all vector $\varepsilon$-Slater minimax solutions* (*strategies*) $\hat{X}_s^{\varepsilon}$ *for game* (2) *is defined,* [1, p. 12].

Everywhere it is supposed that if $\varepsilon \equiv \mathbf{0}_N$, i.e. $\varepsilon$ is zero vector in $\mathbb{R}^N$, then $X$ and $Y$ are compact subsets in $\mathbb{R}^m$, respectively in $\mathbb{R}^n$. In this case we shall refer to (vector) Slater saddle point, Slater maximin solution, etc. [1, p.21, 11-12], instead of (vector) $\varepsilon$-Slater saddle point, $\varepsilon$-Slater maximin solution, etc. as in Definitions 1-3, and the sets $S^{\varepsilon}$, $Y_s^{\varepsilon}(x)$, $X_s^{\varepsilon}(y)$, $\hat{Y}_s^{\varepsilon}$ and $\hat{X}_s^{\varepsilon}$ given in Definitions 1-3, we shall denote by $S$, $Y_s(x)$, $X_s(y)$, $\hat{Y}_s$ and $\hat{X}_s$ respectively.

**2. Formulations and proofs of the results obtained.** By analogy with [5, p.158, 142], it is proved the following

**Lemma 1.** *For each fixed vector $\varepsilon \in \mathbb{R}^N_{\geqq}$, $x \in X$ and $y \in Y$, the sets of Definition 2 $Y_s^{\varepsilon}(x)$ and $X_s^{\varepsilon}(y)$ are non-empty. If $\varepsilon \equiv \mathbf{0}_N$ and $X$ and $Y$ are compact sets, then $Y_s(x)$ and $X_s(y)$ are non-empty and compact sets.*

From Definitions 1-2 the following assertion has been obtained:

**Lemma 2.** *Let us suppose in addition to general assumptions in Section 1, that*

*a) the set $Y_s^{\varepsilon}(x)$ does not depend on $x$, i.e. $Y_s^{\varepsilon}(x) = Y_s^{\varepsilon}$ $\forall x \in X$. Then $S^{\varepsilon} = \{(x, y) \big| x \in X_s^{\varepsilon}(y), \ y \in Y_s^{\varepsilon}\}$,*

*b) the set $X_s^{\varepsilon}(y)$ does not depend on $y$, i.e. $X_s^{\varepsilon}(y) = X_s^{\varepsilon}$ $\forall y \in Y$. Then $S^{\varepsilon} = \{(x, y) \big| x \in X_s^{\varepsilon}, \ y \in Y_s^{\varepsilon}(x)\}$.*

*If $Y_s^{\varepsilon}(x) = Y_s^{\varepsilon}$ $\forall x \in X$ and $X_s^{\varepsilon}(y) = X_s^{\varepsilon}$ $\forall y \in Y$, then $S^{\varepsilon} = X_s^{\varepsilon} \times Y_s^{\varepsilon}$.*

As in [1], we denote by $Fr_{\varepsilon}^{s}M$ $(Fr_{s}^{\varepsilon}M)$ the set of all $\varepsilon$-Slater maximal (minimal) points of the arbitrary set $M \subset \mathbb{R}^N$. Also $\bigcup\limits_{x \in X} MAX{}_{\varepsilon}^{s} \bigcup\limits_{y \in Y} MIN{}_{\varepsilon}^{s} f(x, y)$ represents the set of all the $\varepsilon$-Slater maximins, analogous presentation about the set of all the $\varepsilon$-Slater minimaxes is valid, [1, p. 12].

**Theorem 1.** *For the same assumptions as in Lemma* 2*,*

*a)* $Fr_{\varepsilon}^{s} f(S^{\varepsilon}) = \bigcup\limits_{x \in X} MAX{}_{\varepsilon}^{s} \bigcup\limits_{y \in Y} MIN{}_{\varepsilon}^{s} f(x, y)$ *for* $Y_s^{\varepsilon}(x) = Y_s^{\varepsilon}$ $\forall x \in X$*,*

*b)* $Fr_{s}^{\varepsilon} f(S^{\varepsilon}) = \bigcup\limits_{y \in Y} MIN{}_{\varepsilon}^{s} \bigcup\limits_{x \in X} MAX{}_{\varepsilon}^{s} f(x, y)$ *for* $X_s^{\varepsilon}(y) = X_s^{\varepsilon}$ $\forall y \in Y$*.*

**Proof.** We give the proof only of the first assertion a). We denote

$$M = f(S^{\varepsilon}) = \{f(x, y) \big| x \in X_s^{\varepsilon}(y), \ y \in Y_s^{\varepsilon}\}, \qquad D = \{f(x, y) \big| x \in X, \ y \in Y_s^{\varepsilon}\}.$$

Then

$$(3)\qquad \begin{aligned} A &\overset{\text{def}}{=} \bigcup \underset{x\in X}{MAX}\,{}^s_\varepsilon \bigcup \underset{y\in Y}{MIN}\,{}^s_\varepsilon f(x,y) = \bigcup \underset{x\in X}{MAX}\,{}^s_\varepsilon \bigcup_{y^\varepsilon_s \in Y^\varepsilon_s} f\left(x, y^\varepsilon_s\right) \\ &= \bigcup \{a \in D : a \not< f\left(x, y^\varepsilon_s\right) - \varepsilon,\ \forall x \in X,\ \forall y^\varepsilon_s \in Y^\varepsilon_s\} \\ &= \bigcup \{a \in M : a \not< f\left(x, y^\varepsilon_s\right) - \varepsilon,\ \forall x \in X,\ \forall y^\varepsilon_s \in Y^\varepsilon_s\}\,, \end{aligned}$$

$$(4)\qquad B \overset{\text{def}}{=} Fr^s_\varepsilon f(S^\varepsilon) = \bigcup \{b \in M : b \not< f(x,y) - \varepsilon,\ \forall x \in X^\varepsilon_s(y),\ \forall y \in Y^\varepsilon_s\}\,.$$

From (3) and (4) it follows that $A \subseteq B$.

Now, let us suppose that $\exists b \in B$ and $b \notin A$. Then

$$(5)\qquad b \in M : b < f\left(x^{(1)}, y^{(1)}{}^\varepsilon_s\right) - \varepsilon \text{ for some } x^{(1)} \in X \text{ and } y^{(1)}{}^\varepsilon_s \in Y^\varepsilon_s.$$

We consider the set $X_1 = \{x \in X : b + \delta \leqq f\left(x, y^{(1)}{}^\varepsilon_s\right) - \varepsilon\}$. Then from (5), $\emptyset \neq X_1 \subseteq X$ for "sufficiently small vector" $\delta > \mathbf{0}_N$. When $\varepsilon = \mathbf{0}_N$, $X$ and $X_1$ are compact sets. From Lemma 1 the set $X_1{}^\varepsilon_s\left(y^{(1)}{}^\varepsilon_s\right) \neq \emptyset\ \ \forall \varepsilon \in \mathbb{R}^N_{\geqq}$. Thus let $\hat{x} \in X_1{}^\varepsilon_s\left(y^{(1)}{}^\varepsilon_s\right)$. It is easy to prove, that $\hat{x} \in X^\varepsilon_s\left(y^{(1)}{}^\varepsilon_s\right)$. Hence

$$(6)\qquad b < f\left(\hat{x}, y^{(1)}{}^\varepsilon_s\right) - \varepsilon, \text{ where } \hat{x} \in X^\varepsilon_s\left(y^{(1)}{}^\varepsilon_s\right),\ y^{(1)}{}^\varepsilon_s \in Y^\varepsilon_s.$$

But (6) shows that $b$ cannot belong to set (4), i.e. $b \notin B$. This contradiction is due to the supposition that $\exists b \in B$ such that $b \notin A$, which is not true.

Thus it is proved that $B \subseteq A$ and hence $A \equiv B$. Theorem 1 is proved. $\square$

**Theorem 2.** *Let the components of* (1) $f_j(x,y) = g_j(\varphi_j(x), \psi_j(y))$ *for* $j = 1, \ldots, k$ *and* $f_j(x,y) = \varphi_j(x) - \psi_j(y)$ *for* $j = k+1, \ldots, N$, *(for arbitrary* $k = 0, \ldots, N$*), the vector* $\varepsilon = (0, \ldots, 0, \varepsilon_{k+1}, \ldots, \varepsilon_N) \in \mathbb{R}^N_{\geqq}$ *is such that* $\varepsilon_j \geqq 0$ *are arbitrary fixed numbers for* $\forall j = k+1, \ldots, N$. *Let the bounded functions* $\varphi_j(x) \in C(X)$ *and* $\psi_j(y) \in C(Y)$, $\varphi_j(X) \overset{\text{def}}{=} \{\varphi_j(x) \big| x \in X\} \subset \mathbb{R}$ *and* $\psi_j(Y) \overset{\text{def}}{=} \{\psi_j(y) \big| y \in Y\} \subset \mathbb{R}$ *for* $\forall j = 1, \ldots, N$, *the bounded functions* $g_j(s,t) \in C(\varphi_j(X) \times \psi_j(Y))$ *are strictly monotonous with respect to* $t$ *for fixed* $s$ *(*$s$ *for fixed* $t$*) for* $\forall (s,t) \in (\varphi_j(X), \psi_j(Y))$ *and* $\forall j = 1, \ldots, k$. *Then the set* $Y^\varepsilon_s(x)$ *(*$X^\varepsilon_s(y)$*) does not depend on* $x \in X$ *(*$y \in Y$*). For the game of type (2) thus defined, Lemma 2 and Theorem 1 are valid.*

**Proof.** Let us suppose that $g_j(s,t)$ are strictly increasing with respect to $s$ for fixed $t$ (strictly decreasing with respect to $t$ for fixed $s$) for $\forall (s,t) \in (\varphi_j(X), \psi_j(Y))$ and $\forall j = 1, \ldots, k$. Let $y \in Y$ be an arbitrary point. We shall show that

$$(7)\qquad x^0 \in X^\varepsilon_s(y) \iff x^0 \in X \text{ and } \varphi(x^0) \in Fr^s_\varepsilon \varphi(X),$$

where $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x))$ for $x \in X$ and $\varphi(X) = (\varphi_1(X), \ldots, \varphi_N(X))$. Indeed, from Definition 2,

$$x^0 \in X^\varepsilon_s(y) \iff x_0 \in X \text{ and } f(x^0, y) \not< f(x,y) - \varepsilon \forall x \in X,$$

i.e.

$$(8)\qquad g_{j_0}(\varphi_{j_0}(x^0), \psi_{j_0}(y)) \geqq g_{j_0}(\varphi_{j_0}(x), \psi_{j_0}(y)) \text{ for some } j_0 = 1, \ldots, k,$$

or

(9) $$\varphi_{j_1}(x^0) \geqq \varphi_{j_1}(x) - \varepsilon_{j_1} \text{ for some } j_1 = k+1, \dots, N,$$

where $j_0$ (or $j_1$) depends on $x \in X$. But (8) is valid if and only if $\varphi_{j_0}(x^0) \geqq \varphi_{j_0}(x)$. Therefore (8) or (9) are valid $\iff$ $(\varphi_1(x^0), \dots, \varphi_N(x^0)) \not< (\varphi_1(x), \dots, \varphi_N(x)) - \varepsilon$ for thus defined vector $\varepsilon$ and arbitrary point $x \in X$, which proves (7). Hence the set $X_s^{\varepsilon}(y)$ does not depend on $y \in Y$. The assertion about $Y_s^{\varepsilon}(x)$ is proved by analogy.

In the general case, we replace $g_j(s,t)$, $\varphi_j(x)$ and $\psi_j(y)$ by the functions $\hat{g}_j(s,t) = g_j(\pm s, \pm t)$, $\hat{\varphi}_j(x) = \pm\varphi_j(x)$ and $\hat{\psi}_j(y) = \pm\psi_j(y)$ so that $\hat{g}_j(\hat{\varphi}_j(x), \hat{\psi}_j(y)) \equiv g_j(\varphi_j(x), \psi_j(y)) = f_j(x,y)$ $\forall (x,y) \in X \times Y$ and $\hat{g}_j(s,t)$ are strictly increasing with respect to $s$ for fixed $t$ (strictly decreasing with respect to $t$ for fixed $s$) for $\forall (s,t) \in (\hat{\varphi}_j(X), \hat{\psi}_j(Y))$ and $\forall j = 1, \dots, k$. Thus the considerations are reduced to the previous case. Theorem 2 is proved. $\square$

**Corollary 1.** *Let $f(x,y) = \varphi(x) + \psi(y)$, where $\varphi(x)$ and $\psi(y)$ are vector functions, defined, bounded and continuous for $x \in X$ and $y \in Y$ respectively. Then for $\forall \varepsilon \in \mathbb{R}^N_{\geqq}$, $\forall x \in X$ and $\forall y \in Y$, the sets $Y_s^{\varepsilon}(x)$ and $X_s^{\varepsilon}(y)$ do not depend on $x$ and $y$. For such a game Lemma 2 and Theorem 1 are valid, see* [1, p.81-85].

**3. Examples.**

**Example 1.** *Let $f(x,y) = (xy, x+y)$, where $X = Y = [\delta_0, 1]$, $\delta_0 > 0$ and $\varepsilon > 0$ are sufficiently small numbers, such that* $0 < \delta_0 \leqq \varepsilon < 1/4$ *and $\vec{\varepsilon} = (\varepsilon, \varepsilon)$.*

Let us note that the functional $f(x,y)$ satisfies the conditions of Theorem 2. Using Definitions 1-3 it is proved that: 1) the point $(x, \varepsilon/x)$ is $\vec{\varepsilon}$-Slater saddle point if and only if $x \in [1/2, 1]$ and 2) the point $(x, \varepsilon/x)$ corresponds to $\vec{\varepsilon}$-Slater maximin solution for $\forall\ x \in [\varepsilon, 1]$.

Thus it is obtained that there exists an $\vec{\varepsilon}$-Slater maximin $(\varepsilon, x + \varepsilon/x) \notin f(S^{\varepsilon})$ for some $x \in [\varepsilon, 1/2)$. Thus, for so defined antagonistic game, Theorem 1a) is not valid for so defined vector $\vec{\varepsilon}$.

On the other hand, if we replace the above sets of strategies with the sets $X = Y = [1,2]$, then the sets $Y_s^{\vec{\varepsilon}}(x)$ and $X_s^{\vec{\varepsilon}}(y)$ do not depend on $x$ and $y$ for $\forall x \in X$ and $\forall y \in Y$, i.e. Lemma 2 and Theorem 1 are valid for this case.

**Example 2.** *Let us consider the following bicriterial antagonistic game with quadratic pay-off function $f(x,y) = (f_1(x,y), f_2(x,y))$, where*

$$f_1(x,y) = -x^2 - 2xy + y^2,\ \ f_2(x,y) = -x^2 + xy + y^2 \ \textit{and}\ X = Y = [0,1].$$

Using Definitions 1-3, for this game it is proved that: 1) the sets $Y_s(x) = [0,x]\ \forall x \in X$, $X_s(y) = [0, y/2]\ \forall y \in Y$ and $S = (0,0)$, 2) all the points–pairs of strategies, which correspond to Slater maximin (minimax) solutions are of the form $\{(x,x) | \forall x \in [0,1]\}$ $(\{(y/2, y) | \forall y \in [0,1]\})$.

The considered game is interesting for the fact, that for the scalar functional $f_{(k)} = kf_1 + (1-k)f_2$, $(0,0) \in X \times Y$ is the unique saddle point $\forall k \in [0,1]$. At the same time, the sets of all Slater maximins and minimaxes are of the form $\{(-2x^2, x^2) | \forall x \in [0,1]\}$

and $\{(-y^2/4, 5y^2/4)\big|\forall y \in [0,1]\}$, respectively. Thus the assertions of Theorem 1 are not true for this game.

**Example 3.** *Let us consider the following quadratic antagonistic game of the previous type, for which*

$$f(x,y) = (-x^2 - xy + y^2, -x^2 + xy + y^2), \ \textit{where } X = Y = [0,1].$$

By analogy, it is proved that the sets $S = (0,0)$, $\hat{Y}_s = \hat{X}_s = \{0\}$, $Y_s(x) = [0, x/2]$ $\forall x \in X$ and $X_s(y) = [0, y/2]$ $\forall y \in Y$, i.e. this game does not satisfy the conditions of Theorem 1. But the assertions of Theorem 1 are true for this game.

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## ВЕКТОРНИ СЕДЛОВИ ТОЧКИ И МАКСИМИНИ ПО СЛЕЙТЪР ПРИ НЯКОИ ВИДОВЕ МНОГОКРИТЕРИАЛНИ АНТАГОНИСТИЧНИ ИГРИ

**Дико Моис Суружон**

Лема 2 и Теорема 1 на представената работа обобщават някои резултати от [1, с. 81–91], отнасящи се за многокритериални антагонистични игри със сепарабелна платежна функция. Теорема 2 намира един по-широк клас игри, включващ сепарабелните, за които са в сила Лема 2 и Теорема 1. Като следствие тук се получават резултатите от [1]. Пример 1 показва, че Теорема 2 не може да бъде обобщена за произволни вектори $\varepsilon$ с неотрицателни компоненти. Теорема 1 не може да бъде обобщена за класа на квадратичните многокритериални антагонистични игри, разгледани в Пример 2, даже и в случая, когато $\varepsilon$ е нулев вектор. Подобна квадратична игра, която не удовлетворява условията на Лема 2 и Теорема 1, е разгледана в Пример 3, но за нея са верни твърденията на Теорема 1.