# THE BOND MARKET WITH STOCHASTIC VOLATILITY IN HIGH LEVEL OF INFLATION 

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#### Abstract

We consider a continuous-time arbitrage free model of the term structure of interest rates. The inflation is given by the process of price level. There are two factors in the model. We assume the existence of futures contracts written on the instantaneous variances of the zero-coupon bond price. These contracts can be replicated by a trading strategy in interest rate futures in real units.


1. Introduction. The relationship between interest rates on default-free bonds of different maturities has been the topic of many papers. The most notable are the works of Vasicek [7], Richard [5] and Björk [2]. This paper is a survey of the general theory of term structure of interest rate process.

We consider a continuous-time arbitrage free model of the term structure of interest rates. The inflation is given by the process of price level. There are two factors in the model. The process of zero-coupon bond price is driven by the first factor only. The process of price level is affected by the second factor.

As in the paper [1], we derive the risk neutral dynamics of both factors by arbitrage free arguments. We assume the existence of futures contracts written on the instantaneous variances of the zero-coupon bond price. These contracts can be replicated by a trading strategy in interest rate futures in real units.
2. Definitions and preliminary results. All processes are assumed to be defined on the complete probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right), P\right)$ and adapted to the filtration $\left(\mathcal{F}_{t}\right)$. The notation $E_{t}(\cdot)$ will be used throughout instead of $E\left(\cdot \mid \mathcal{F}_{t}\right)$.

A zero-coupon bond (or discount bond) of maturity $T$, also called $T$-bond, is a contract which guarantees the holder 1 unit of cash to be paid on the date $T$ in the future. The price at time $t$ of a bond with maturity $T$ is denoted by $B(t, T)$.

Suppose we are standing at time $t<T$.
Definition 1. The yield-to-maturity of a zero-coupon bond maturing at time $T$ is an adapted process $R(t, T)$ defined by

$$
B(t, T) \exp [(T-t) R(t, T)]=1,
$$

or

$$
\begin{equation*}
R(t, T)=-\frac{1}{T-t} \ln B(t, T), \quad t \in[0, T) \tag{1}
\end{equation*}
$$

The rates $R(t, T)$ considered as a function of $T$ is known as the term structure of interest rates (also as the yield curve) at time $t$, see Vasicek [7].

Definition 2. The yield on the currently maturing bond is called short-term interest rate or spot interest rate at time $t$ :

$$
r(t)=R(t, t)
$$

It is the rate of return investors can earn over next very short time interval. Actually $r(t)=\lim _{T \rightarrow t} R(t, T)$. Applying L'Hopital's rule to (1) we find that

$$
r(t)=-\left.\frac{\partial \ln B(t, T)}{\partial T}\right|_{T=t}
$$

Definition 3. Consider an investor who now holds a bond maturing at time $S$. Between time $S$ and time $T>S$ he would earn the forward interest rate $f(t, S, T)$, defined by

$$
B(t, S)=B(t, T) \exp [(T-S) f(t, S, T)]
$$

or

$$
\begin{equation*}
f(t, S, T)=-\frac{\ln B(t, T)-\ln B(t, S)}{T-S} \tag{2}
\end{equation*}
$$

Definition 4. The instantaneous forward rate $f(t, T)$ is defined by the formula

$$
\begin{equation*}
f(t, T)=\lim _{S \rightarrow T} f(t, S, T) \tag{3}
\end{equation*}
$$

Combining (2) and (3) we get

$$
f(t, T)=-\frac{\partial \ln B(t, T)}{\partial T}
$$

Actually $f(t, T)$ should be interpreted as the interest rate over the infinitesimal time interval $[T, T+d T]$ as seen from time $t$ and $f(t, T)=f(t, T, T)$.

The definitions give the following useful representation

$$
B(t, T)=\exp \left(-\int_{t}^{T} f(t, u) d u\right), \quad t \leq s \leq T
$$

This means that the shape of the current bond price curve is determined by all instantaneous forward rates.

Let us return back to the spot rate. It is actually the instantaneous interest rate for risk-free borrowing or lending prevailing at time $t$ over the infinitesimal time interval $[t, t+d t]$ and $r(t)=f(t, t)$. Now we can introduce an adapted process $B(t)$ of finite variation with continuous sample paths, given by

$$
B(t)=\exp \left(\int_{0}^{t} r(u) d u\right), \quad t \in[0, T] .
$$

For almost all $\omega \in \Omega$, the function $B(t)=B(t, \omega)$ solves the differential equation

$$
d B(t)=B(t) r(t) d t, \quad r(t)>0 .
$$

The process $B(t)$ is referred to as a money market account or an accumulation factor. Intuitively, $B(t)$ represents the amount of cash accumulated up to time $t$ by starting with one unit of cash at time 0 , and continually rolling over a bond with infinitesimal time to maturity.

Definition 5. A family $B(t, T), t \leq T \leq T^{*}$, of adapted processes is called an arbitrage-free family of bond prices relative to $r$ if the following conditions are satisfied:
(i) $B(T, T)=1$ for every $T \in\left[0, T^{*}\right]$, and
(ii) there exists a probability measure $Q$ on $\left(\Omega, \mathcal{F}_{T^{*}}\right)$ equivalent to $P$, and such that for any maturity $T \in\left[0, T^{*}\right]$, the relative bond price

$$
\bar{B}(t, T)=\frac{B(t, T)}{B(t)}, \quad t \in[0, T]
$$

is a martingale under $Q$.
The probability measure $Q$ is called a martingale measure for the family $B(t, T)$ relative to $r$.

Now we can give the following useful result (see Rogers [6]):
Lemma. For any martingale measure $Q$ of an arbitrage-free family of bond prices, we have

$$
B(t, T)=E_{t}^{Q}\left(e^{-\int_{t}^{T} r(u) d u}\right), \quad t \in[0, T] .
$$

3. The model. We consider a continuous-time economy where the zero-coupon bonds $\{B(t, T)\}_{t \leq T}$ are traded for all maturities. Absence of arbitrage opportunity implies the existence of an equivalent probability measure $Q$, under which the discounted prices are martingales. Let us suppose that the measure $Q$ exists and the bond price evolve as

$$
\begin{equation*}
d B(t, T)=B(t, T)\left[r(t) d t+\sigma_{B}(t, T) d W_{1}^{Q}(t)\right] \tag{4}
\end{equation*}
$$

where $W_{1}^{Q}$ is a one dimensional Wiener process under $Q$. The coefficient $r(t)$ in (4) is the return on risk-free bond, $\sigma_{B}(t, T)$ is nonnegative bounded volatility coefficient, which in fact is the standard deviation of the process. The variance $\sigma_{B}^{2}(t, T)$ is often studied as a measure of risk.

By the definition of the zero coupon bonds we have the following boundary condition:

$$
B(T, T)=1
$$

We impose $\sigma_{B}(T, T)=0$. We will study the dynamics of the zero coupon bond in the presence of high level of inflation. It is important to see the dynamics of the variance in one future time period $[t, T]$.

We assume that there exist futures contracts on the instantaneous variance of the zero coupon bonds. Such a contract is non realistic. It is not traded on todays financial
markets. But the interpretation is useful and it can be perfectly replicated by trading in futures on yields.

We define $V(t, s, T), t \leq s \leq T$, to be the time $t$ continuously marked to market futures price for delivery of $\sigma_{B}^{2}(s, T)$ at time $s$. The process $V(t, s, T)$ is a martingale under the measure $Q$ and $V(t, T, T)=0$.

Let us describe the model in real units. We shall apply the method used of [4]. The rate of inflation is the rate of growth of the price level, $P(t)$, which is the money price of a unit of consumption good,

$$
d P(t)=P(t)[\pi(t) d t+\sigma(t) d Z(t)]
$$

In the above equation the coefficient $\pi(t)$ can be interpreted as an inflation coefficient and $\sigma^{2}(t)$ as an inflation risk. We suppose that the Brownian motion $Z(t)$ is independent of $W_{1}^{Q}(t)$. In the market defined above we choose as the "numéraire" the process $P(t)$ (see [3] and [4]), i.e. the prices of all other assets will be evaluated in units of $P(t)$. The process $\bar{B}(t, T)=\frac{B(t, T)}{P(t)}$ represents the price of the zero-coupon bond in real units. Using the Ito's formula we obtain the following evolution equation

$$
\begin{equation*}
d \bar{B}(t, T)=\bar{B}(t, T)\left[\left(r(t)-\pi(t)+\sigma^{2}(t)\right) d t+\sigma_{B}(t, T) d W_{1}^{Q}(t)-\sigma(t) d Z(t)\right] \tag{5}
\end{equation*}
$$

We can specify the volatility coefficient of $\bar{B}(t, T)$ as

$$
\sigma(t, T)=\sqrt{\left(\sigma_{B}(t, T)\right)^{2}+(-\sigma(t))^{2}}
$$

and define the following Brownian motion

$$
W(t)=\int_{0}^{t} \frac{\sigma_{B}(u, T) d W_{1}^{Q}(u)-\sigma(u) d Z(u)}{\sigma(u, T)} .
$$

So, the equation (5) can be written as

$$
\begin{equation*}
d \bar{B}(t, T)=\bar{B}(t, T)\left[\left(r(t)-\pi(t)+\sigma^{2}(t)\right) d t+\sigma(t, T) d W(t)\right], \quad t \geq 0 \tag{6}
\end{equation*}
$$

4. The Hedging strategy. In the previous section we assumed the existence of a futures contract on the instantaneous volatility of bond price.

Let us consider now a futures contract on the continuously compounded yield $\bar{R}(t, T)$ as defined in (1), but in real units.

The no arbitrage condition involves that the futures price for such a contract is

$$
Y(t, s, T)=E_{t}^{Q}[\bar{R}(s, T)] .
$$

Integrating (6) we get
$\ln \bar{B}(t, T)=\ln \bar{B}(0, T)+\int_{0}^{t}\left[r(u)-\pi(u)-\frac{\sigma_{B}^{2}(u, T)}{2}+\frac{\sigma^{2}(u)}{2}\right] d u+\int_{0}^{t} \sigma(u, T) d W(u)$.
Hence, taking the expectation of this equation with respect the measure $Q$ and using the relation (1), we get the following

$$
Y(0, t, T)(T-t)=-\ln \bar{B}(0, T)-E_{0}^{Q} \int_{0}^{t}\left[r(u)-\pi(u)-\frac{\sigma_{B}^{2}(u, T)}{2}+\frac{\sigma^{2}(u)}{2}\right] d u
$$

Differentiation in the above equation leads to

$$
\frac{\partial}{\partial t}\{(T-t) Y(0, t, T)\}=-E_{0}^{Q}[r(t)-\pi(t)]+\frac{1}{2} E_{0}^{Q}\left[\sigma_{B}^{2}(t, T)\right]-\frac{1}{2} E_{0}^{Q} \sigma^{2}(t)
$$

Rearranging the terms we get the following
(7) $E_{0}^{Q}\left[\sigma_{B}^{2}(t, T)\right]=2(T-t) \frac{\partial}{\partial t} Y(0, t, T)-2 Y(0, t, T)+2 E_{0}^{Q}[r(t)]-2 E_{0}^{Q}\left[\pi(t)-\frac{\sigma^{2}(t)}{2}\right]$.

Noting that according to the definition of the $Y(t, s, T)$, for every $T>0$ we have the following

$$
\lim _{t \rightarrow T} Y(0, t, T)=E_{0}^{Q}\left[r(T)-\pi(T)+\frac{1}{2} \sigma^{2}(T)\right]=Y(0, T, T)
$$

So, the equation (7) implies that

$$
V(0, t, T)=2(T-t) \frac{\partial}{\partial t} Y(0, t, T)-2 Y(0, t, T)+2 Y(0, t, t)
$$

This is consistent with the condition $\sigma_{B}(T, T)=0$. We can see also that the futures contract we used in the previous section $V(t, s, T)$ is equivalent to the futures contracts on continuously compounded real yields.

In order to replicate the $V(0, t, T)$ contract we would have to take a static hedging position consisting of:

1. A long position in $\frac{2(T-t)}{\Delta t}$ futures contracts on the real yield between time $t+\Delta t$ and $T$;
2. A short position in $2\left(1+\frac{T-t}{\Delta t}\right)$ futures contracts on the real yield between time $t$ and $T$;
3. A long position in futures contracts maturing at time $t$ on the instantaneous real yield.

So, the only contract requires to replicate our $V(t, s, T)$ contract is a futures on a yield, a market which already exists. This type of approach might be easy to apply to the pricing of fixed income derivatives.

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## ПАЗАР НА ОБЛИГАЦИИ ПРИ УСЛОВИЯТА НА ВИСОКА ИНФЛАЦИЯ

## Леда Димитрова Минкова

Работата сьдьржа обзор вьрху времевата структура на лихвата. Инфлацията се изразява чрез процеса на ценовото равнище. Разглежда се двуфакторен модел на пазара на облигации с нулев купон. Предполага се сьществуване на бьдещи контракти вьрху моментните дисперсии на цените на облигации с нулев купон. Такива контракти на пазара не сьществуват. Те могат да се неутрализират чрез подходящо избрана стратегия с реално осьществими бьдещи контракти вьрху реалната лихва. Това дава вьзможност за прогнозиране разсейването на цените на облигациите

