## LAGRANGE'S INTERPOLATION FORMULA IN OLYMPIAD PROBLEMS

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The paper considers some problems that can be treated by Lagrange's interpolation formula.

The main theoretical facts, which are used in the paper, are formulated in the following two theorems

Theorem 1. Let $f(x)=a_{0} x^{n}+a_{1} x^{n-1}+\ldots+a_{n-1} x+a_{n}, f(x) \in \mathbb{R}[x], a_{0} \neq 0$, $g(x)=b_{0} x^{n}+b_{1} x^{n-1}+\ldots+b_{n-1} x+b_{n}, g(x) \in \mathbb{R}[x], b_{0}$ may be 0 as $g(x)$ may be of degree less than $n$. Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n+1}$ be distinct numbers, such that $f\left(\alpha_{i}\right)=g\left(\alpha_{i}\right)$ for $i=1,2, \ldots, n+1$. Then $f(x) \equiv g(x)$.

Theorem 2 (Lagrange). Let $n \in \mathbb{N}$ and $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n+1}$ be distinct numbers, $\beta_{1}, \beta_{2}, \ldots, \beta_{n+1}$ be arbitrary numbers. Then there exists an unique polynomial $f(x)$, such that deg $f \leq n$ and $f\left(\alpha_{i}\right)=\beta_{i}$ for $i=1,2, \ldots, n+1$.

This polynomial is

$$
f(x)=\sum_{i=1}^{n+1} \beta_{i} \prod_{j \neq i, j=1}^{n+1} \frac{x-\alpha_{j}}{\alpha_{i}-\alpha_{j}}
$$

known as Lagrange's interpolation polynomial.
In the sequel some olympiad problems are discussed.
The first problem is taken from the shortlist of the International Mathematical Olympiad (IMO) in 1981 in the United States.

Problem 1. A polynomial $f(x)$, deg $f=n$, satisfies $f(k)=\frac{1}{\binom{n+1}{k}}$ for $k=0,1, \ldots, n$. Find $f(n+1)$.

Solution. Applying Lagrange's interpolation formula directly, we get

$$
\begin{aligned}
f(x) & =\sum_{k=0}^{n} \frac{1}{\binom{n+1}{k}} \prod_{i=0, i \neq k}^{n} \frac{x-i}{k-i}=\sum_{k=0}^{n} \frac{\prod_{i=0, i \neq k}^{n}(x-i)}{\binom{n+1}{k}(-1)^{n-k}(n-k)!k!}= \\
& =\sum_{k=0}^{n}(-1)^{n-k} \frac{n+1-k}{(n+1)!} \prod_{i=0, i \neq k}^{n}(x-i) .
\end{aligned}
$$

Setting $x=n+1$ we get

$$
\begin{aligned}
f(n+1) & =\sum_{k=0}^{n}(-1)^{n-k} \frac{n+1-k}{(n+1)!} \prod_{i=0, i \neq k}^{n}(n+1-i)= \\
& =\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n+1)!} \prod_{i=0}^{n}(n+1-i)=\sum_{k=0}^{n}(-1)^{n-k}
\end{aligned}
$$

Thus,

$$
f(n+1)= \begin{cases}0, & \text { when } n \text { is odd } \\ 1, & \text { when } n \text { is even }\end{cases}
$$

The next problem is from the shortlist of the IMO'79 in Great Britain.
Problem 2. A polynomial $f(x)$, deg $f \leq 2 n$, satisfies $|f(k)| \leq 1$ for every integer $k \in[-n, n]$. Prove that for an arbitrary number $x \in[-n, n]$ the inequality $|f(x)| \leq 2^{2 n}$ holds true.

Solution. According to Lagrange's interpolation formula and Theorem 2

$$
f(x)=\sum_{k=-n}^{n} f(k) \prod_{i \neq k, i=-n}^{n} \frac{x-i}{k-i}
$$

Since $|f(k)| \leq 1$ for $k=-n,-n+1, \ldots, n$, then

$$
|f(x)| \leq \sum_{k=-n}^{n}|f(k)| \prod_{i \neq k, i=-n}^{n} \frac{|x-i|}{|k-i|} \leq \sum_{k=-n}^{n} \prod_{i \neq k, i=-n}^{n} \frac{|x-i|}{|k-i|}
$$

We shall prove that for every real number $x \in[-n, n]$, we have

$$
\prod_{i \neq k, i=-n}^{n}|x-i| \leq(2 n)!
$$

When $x \geq k$, we have:

$$
\begin{aligned}
\prod_{i \neq k, i=-n}^{n}|x-i| & =(|x-(k+1)| \ldots|x-n|)(|x-(k-1)| \ldots|x+n|) \\
\leq & (n-k)!((n-k+1) \ldots(2 n))=(2 n)!
\end{aligned}
$$

In the case $x<k$ we prove the assertion analogously.
Using the last result, we get

$$
\prod_{i \neq k, i=-n}^{n} \frac{|x-i|}{|k-i|} \leq(2 n)!\prod_{i \neq k, i=-n}^{n} \frac{1}{|k-i|}=(2 n)!\frac{1}{(k+n)!(n-k)!}
$$

Then, using the well-known fact that $\sum_{j=1}^{n}\binom{n}{j}=2^{n}$, we get

$$
|f(x)| \leq \sum_{k=-n}^{n} \frac{(2 n)!}{(k+n)!(n-k)!}=\sum_{k=0}^{2 n} \frac{(2 n)!}{k!(2 n-k)!}=\sum_{k=0}^{2 n}\binom{2 n}{k}=2^{2 n}
$$

which ends the proof.

The third problem is from another shortlist - that of the IMO'77 in Yugoslavia
Problem 3. Integers $x_{0}<x_{1}<\ldots<x_{n}$ and a polynomial $f(x)=x^{n}+a_{1} x^{n-1}+$ $\ldots+a_{n-1} x+a_{n}, f(x) \in \mathbb{R}[x]$, are given. Prove that there exists $i \in\{0,1, \ldots n\}$, such $\cdots+a_{n-1} x+a_{n}$,
that $\left|f\left(x_{i}\right)\right| \geq \frac{n!}{2^{n}}$.

Solution. By Lagrange's interpolation formula and Theorem 2 we get:

$$
f(x)=\sum_{i=0}^{n} f\left(x_{i}\right) \prod_{j \neq i, j=0}^{n} \frac{x-x_{j}}{x_{i}-x_{j}}
$$

Let us assume that the assertion of the problem is not true, i.e. $\left|f\left(x_{i}\right)\right|<\frac{n!}{2^{n}}$ for $i=0,1, \ldots, n$. Since the senior coefficient of the polynomial $f(x)$ is equal to the sum of the senior coefficients of the products $f\left(x_{i}\right) \prod_{j \neq i, j=0}^{n} \frac{x-x_{j}}{x_{i}-x_{j}}$, then the absolute value of the senior coefficient of $f(x)$ verifies

$$
\begin{gathered}
\left|\sum_{i=0}^{n} f\left(x_{i}\right) \prod_{j \neq i, j=0}^{n} \frac{1}{x_{i}-x_{j}}\right|<\sum_{i=0}^{n} \frac{n!}{2^{n}} \prod_{j \neq i, j=0}^{n} \frac{1}{\left|x_{i}-x_{j}\right|} \leq \\
\sum_{i=0}^{n} \frac{n!}{2^{n}} \frac{1}{\prod_{j=0}^{i-1}(i-j)} \frac{1}{\prod_{j=i+1}^{n}(j-i)}=\frac{1}{2^{n}} \sum_{j=0}^{n}\binom{n}{j}=1 .
\end{gathered}
$$

(We have used the fact that $x_{i}-x_{j} \geq i-j(i>j)$ since $x_{0}, x_{1}, \ldots, x_{n}$ are integers satisfying $x_{0}<x_{1}<\ldots<x_{n}$.)

But the senior coefficient of $f(x)$ is 1 , which is a contradiction, thus proving the assertion of the problem.

The following problem is taken from the shortlist of the IMO'97 in Argentina.
Problem 4. Let $p$ be a prime number and let $f(x)$ be a polynomial with $\operatorname{deg} f=d$ and $f(x) \in \mathbb{Z}[x]$, satisfying the following two conditions:
(i) $f(0)=0, f(1)=1$;
(ii) for every positive integer $n$, the remainder of $f(n)$ divided by $p$ is either 0 or 1 .

Prove that $d \geq p-1$.
Solution. Let us assume that $d \leq p-2$. Then, according to Theorem 2, $f(x)$ is determined by its values in $0,1, \ldots p-2$. By Lagrange's interpolation formula, we have

$$
f(x)=\sum_{k=0}^{p-2} f(k) \frac{x(x-1) \ldots(x-k+1)(x-k-1) \ldots(x-p+2)}{k!(-1)^{p-k}(p-k+2)!} .
$$

Setting $x=p-1$ in this formula, we get

$$
f(p-1)=\sum_{k=0}^{p-2} f(k) \frac{(p-1) \ldots(p-k)}{k!(-1)^{p-k}}=\sum_{k=0}^{p-2} f(k)(-1)^{p-k}\binom{p-1}{k}
$$

Now, a simple induction on $k$ shows that if $p$ is a prime and $0 \leq k \leq p-1$, then $\binom{p-1}{k} \equiv(-1)^{k}(\bmod p)$. This is certainly true for $k=0$. Then, if we assume that
for an integer $k$ with $1 \leq k \leq p-1$ we have $\binom{p-1}{k-1} \equiv(-1)^{k-1}(\bmod p)$, using the well-known facts that $\binom{p-1}{k}=\binom{p}{k}-\binom{p-1}{k-1}$ and that $\binom{p}{k}$ is divisible by $p$, we get $\binom{p-1}{k} \equiv(-1)^{k}(\bmod p)$, which completes the induction argument.

Using the last relation, we get that

$$
f(p-1) \equiv(-1)^{p} \sum_{k=0}^{p-2} f(k)(\bmod p) .
$$

This can be rewritten as $f(0)+f(1)+\ldots+f(p-1) \equiv 0(\bmod p)$. Setting $S(f)=$ $f(0)+f(1)+\ldots+f(p-1)$, we get $S(f) \equiv 0(\bmod p)$. Now, it suffices to show that the above relation contradicts to conditions $(i)$ and (ii). It follows from (ii) that $S(f) \equiv k(\bmod p)$, where $k$ denotes the number of those $n \in\{0,1, \ldots, p-1\}$ for which $f(n) \equiv 1(\bmod p)$. On the other hand $(i)$ implies that $k \leq p-1$ and $k \geq 1$. Hence, $S(f) \not \equiv 0(\bmod p)$. We get a contradiction, which proves the assertion of the problem.

The last problem is from the United States of America Mathematical Olympiad in 1995.

Problem 5. Let $q_{0}, q_{1}, q_{2}, \ldots$ be an infinite sequence of integers with the following properties:
(i) $(m-n)$ divides $\left(q_{m}-q_{n}\right)$ for arbitrary $m>n \geq 0$;
(ii) there exists a polynomial $P(x), P(x) \in \mathbb{R}[x]$, such that $\left|q_{n}\right|<P(n)$ for every $n$.

Prove that there exists a polynomial $Q(x), Q(x) \in \mathbb{Q}[x]$, such that $q_{n}=Q(n)$ for every $n$.

Solution. Let $\operatorname{deg} P=d$. According to Lagrange's interpolation formula there exists a polynomial $Q(x)$, such that $\operatorname{deg} Q \leq d, Q(x) \in \mathbb{Q}[x]$ and $Q(i)=q_{i}$ for $i=0,1, \ldots, d$, i.e.

$$
Q(x)=\sum_{i=0}^{d} q_{i} \prod_{j \neq i, j=0}^{d} \frac{x-j}{i-j} .
$$

We shall prove that $q_{n}=Q(n)$ for every $n$.
Let $k$ be the least common multiple of the coefficients of $Q(x)$ and let $r_{n}=k(Q(n)-$ $\left.q_{n}\right)$. Then we have $r_{0}=r_{1}=\ldots=r_{d}=0$.

Since $\left|r_{n}\right| \leq k\left(|Q(n)|+\left|q_{n}\right|\right)<k(|Q(n)|+P(n))$, then there exists a polynomial $R(x)$ such that $\operatorname{deg} R \leq d$ and $\left|r_{n}\right|<R(n)$ for every $n$.

Since $(m-n) /\left(q_{m}-q_{n}\right)$ for arbitrary $m>n \geq 0$, then $(m-n) /\left(r_{m}-r_{n}\right)$ for arbitrary $m>n \geq 0$ because $(m-n)(Q(m)-Q(n))$. This implies that for every $n>d$ and $0 \leq i \leq d$ we have $(n-i) /\left(r_{n}-r_{i}\right)=r_{n}$.

Hence, the least common multiple of $n, n-1, \ldots, n-d$ divides $r_{n}$.
To continue the solution we need the following
Lemma. For arbitrary positive integers $a_{1}, a_{2}, \ldots, a_{m}$ the following inequality holds true

$$
\text { l.c. } m .\left(a_{1}, a_{2}, \ldots, a_{m}\right) \geq \frac{a_{1} a_{2} \ldots a_{m}}{\prod_{1 \leq i<j \leq m} g . c . d .\left(a_{i}, a_{j}\right)}
$$

where l.c.m. stands for the least common multiple and g.c.d. for the greatest common divisor.

Proof. Let $e_{i}$ be the power of a certain prime number $p$ in the canonical representation of $a_{i}$ for $i=1,2, \ldots, m$. Then, the power of $p$ in the canonical representation of l.c.m. $\left(a_{1}, a_{2}, \ldots, a_{m}\right)$ is $\max \left(e_{1}, e_{2}, \ldots, e_{m}\right)$, whereas its power in the canonical representation of $\frac{a_{1} a_{2} \ldots a_{m}}{\prod_{1 \leq i<j \leq m} g . c . \text { d. }\left(a_{i}, a_{j}\right)}$ is $e_{1}+e_{2}+\ldots+e_{m}-\sum_{1 \leq i<j \leq m} \min \left(e_{i}, e_{j}\right)$. So, it suffices to prove that $\max \left(e_{1}, e_{2}, \ldots, e_{m}\right) \geq e_{1}+e_{2}+\ldots+e_{m}-\sum_{1 \leq i<j \leq m} \min \left(e_{i}, e_{j}\right)$.

The last inequality holds true as in the sum of the right-hand side every $e_{i}$, excluding $e_{k}=\max \left(e_{1}, e_{2}, \ldots, e_{m}\right)$, appears as a result of the couples $\left(e_{k}, e_{i}\right)$ and there are also other terms in that sum, resulting from the couples $\left(e_{i}, e_{j}\right)$ where $i, j \neq k$. So, the last inequality holds true clearly.

## Now, let us go back to the problem.

We will prove that there exists a number $N$ such that the inequality l.c.m. $(n, n-$ $1, \ldots, n-d)>R(n)$ holds true for every $n \geq N$. We have

$$
\prod_{0 \leq i<j \leq d} g . c . d .(n-i, n-j)=\prod_{0 \leq i<j \leq d} g . c . d .(n-i, j-i) \leq \prod_{0 \leq i<j \leq d}(i-j)=A
$$

So, applying the Lemma and the last result for the numbers $n, n-1, \ldots, n-d$, we get:

$$
\text { l.c.m. }(n, n-1, \ldots, n-d) \geq \frac{n(n-1) \ldots(n-d)}{\prod_{0 \leq i<j \leq d} g . c . d .(n-i, n-j)} \geq \frac{n(n-1) \ldots(n-d)}{A} .
$$

Since the right-hand side of the last inequality is a polynomial of degree $d+1$, whereas $\operatorname{deg} R \leq d$, then for sufficiently large $n$ we have

$$
\frac{n(n-1) \ldots(n-d)}{A}>R(n)
$$

But since l.c.m. $(n, n-1, \ldots, n-d)$ divides $r_{n}$ and l.c.m. $(n, n-1, \ldots, n-d)>$ $R(n)>\left|r_{n}\right|$, then $r_{n}=0$ for $n \geq N$. Since for arbitrary $m \geq N$ and $n<N$ we have that $(m-n)$ divides $\left(r_{m}-r_{n}\right)=-r_{n}$, then $r_{n}=0$ for every $n<N$. Hence, $r_{n}=0$ for every $n$, which means that $q_{n}=Q(n)$ for every $n$ and we are done.

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## ИНТЕРПОЛАЦИОННАТА ФОРМУЛА НА ЛАГРАНЖ В ЗАДАЧИ ОТ ОЛИМПИАДИ

## Цено Веселинов Целков, Николай Величков Андреев

В статията се разглеждат някои задачи, които могат да бъдат решавани с помощта на интерполационната формула на Лагранж.

