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CELLULAR NEURAL NETWORKS – DYNAMICS AND COMPLEXITY*

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In this paper, Cellular Neural Networks (CNNs) are presented. Since their invention in 1988, the investigation of CNNs has envolved to cover a very broad class of problems and frameworks. Many researchers have made significant contributions to the study of CNN phenomena using different mathematical tools. CNN is simply an analogue dynamic processor array, made of cells, which contain linear capacitors, linear resistors, linear and nonlinear controlled sources. In the paper, a survey of the main types of dynamic equations describing CNNs is made. CNN with hysteresis in the feedback circuit is studied here. Bifurcation, periodic solutions and chaos are proved for some classes of CNN.

One of the most interesting aspects of the world is that it can be considered to be made up of patterns. It is characterized by the order of the elements of which it is made rather than by the intrinsic nature of these elements.

Norbert Wiener

1. Introduction to the Cellular Neural Network paradigm. Many phenomena with complex patterns and structures are widely observed in the nature. For instance, how does the leopard get its spots, or how does the zebra get its stripes, or how does the fingerprint get its patterns? These phenomena are some manifestations of a multidisciplinary paradigm called emergence or complexity. They share a common unifying principle of dynamic arrays, namely, interconnections of a sufficiently large number of simple dynamic units can exibit extremely complex and self-organizing behaviors.

The invention, called Cellular Neural Network (CNN), is due to L. Chua and L. Yang [1, 2] in 1988. Many complex computational problems can be formulated as well-defined tasks where the signal values are placed on a regular geometric 2-D or 3-D grid, and the direct interactions between signal values are limited within a finite local neighborhood. CNN is an analog dynamic processor array which reflects just this property: the processing elements interact directly within a finite local neighborhood.

The concept of CNN is based on some aspects of neurobiology and adapted to integrated circuits. For example, in the brain, the active medium is provided by a sheet-like array of massively interconnected excitable neurons whose energy comes from the burning of glucose with oxygen. In cellular neural networks the active medium is provided by

*2000 Math. Subj. Classification codes: 92B20, 37N25, 34C55, 34C23, 34C25 the local interconnections of active cells, whose building blocks include active nonlinear devices (e.g., CMOS transistors) powered by DC batteries.

Let us consider a two-dimensional grid with 3×3 neighborhood system, as it is shown in Fig.1.

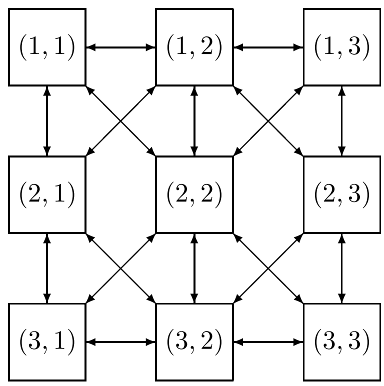

Fig. 1.

The squares are the circuit units – cells $C(i, j)$, and the links between the cells indicate that there are interactions between linked cells. One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We shall give two general definitions of a CNN which follows the original one [1, 2]:

Definition 1.1. *The CNN is a*

a). 2-, 3-, or n-dimensional array of

b). mainly identical dynamical systems, called cells, which satisfies two properties:

c). most interactions are local within a finite radius r, and

d). all state variables are continuous valued signals.

Definition 1.2. *A cellular neural network is a high dimensional dynamic nonlinear circuit composed by locally coupled, spatially recurrent circuit units called cells. The resulting net may have any architecture, including rectangular, hexagonal, toroidal, spherical and so on. An $M \times M$ CNN is defined mathematically by four specifications:*

1). CNN cell dynamics;

2). CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells;

3). Boundary conditions;

4). Initial conditions.

Remark 1.1. The space variable is always discretized and the time variable t may be continuous or discrete.

Remark 1.2. The interconnection between cells is usualy represented by the cloning template which may be a nonlinear function of state x, output y, and input u of each cell $C(i,j)$, within the neighborhood N_r of radius r [1]:

$$N_r(i,j) = \{C(k,l)|max\{|k-i|, |l-j|\} \leq r, 1 \leq k \leq M, 1 \leq l \leq M\}.$$

Moreover, the cloning template has geometrical meaning which we can be exploited to provide the geometric insights and simple design methods.

2. The main types of differential equations describing CNNs. Suppose for simplicity that the processing elements of a CNN are arranged on a 2- dimensional (2-D) grid (Fig.1). Then the dynamics of a CNN, in general, can be described by:

$$\begin{aligned} \dot{x}_{ij}(t) &= -x_{ij}(t) + \sum_{C(kl)\in N_r(ij)} \tilde{A}_{ij,kl}(y_{kl}(t), y_{ij}(t)) + \\ &+ \sum_{C(kl)\in N_r(ij)} \tilde{B}_{ij,kl}(u_{kl}, u_{ij}) + I_{ij}, \end{aligned} \tag{2.1}$$

$$y_{ij}(t) = f(x_{ij}), \tag{2.2}$$

$$1 \leq i \leq M, \qquad 1 \leq j \leq M,$$

x_{ij}, y_{ij}, u_{ij} refer to the state, output and input voltage of a cell $C(i,j)$; $C(ij)$ refers to a grid point associated with a cell on the 2-D grid, $C(kl) \in N_r(ij)$ is a grid point (cell) in the neighborhood within a radius r of the cell $C(ij)$, I_{ij} is an independent current sourse. $\tilde{A}$ and $\tilde{B}$ are nonlinear cloning templates, which specify the interactions between each cell and all its neighbor cells in terms of their input, state, and output variables. In [6,8] the templates are considered to be in the following more general form:

$$\tilde{A} = \begin{bmatrix} 0 & p_1 & 0 \\ p_2 & 2 & p_2 \\ 0 & p_1 & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ p_3 & 1 & p_3 \\ 0 & 0 & 0 \end{bmatrix},$$

with $p_1 = c_1 y_{kl} y_{ij}$, $p_2 = c_2[exp(y_{kl} - 1]$, $p_3 = c_3(u_{kl} - u_{ij})$ in order to assure the stability of the nonlinear CNN. Moreover, $\tilde{A}$ and $\tilde{B}$ are called in [1,2] feedback and control operators.

Some useful output functions f are:

– piece-wise linear sigmoid* function [1,2]:

$$f(x_{ij}) = \frac{1}{2}(|x_{ij} - 1| - |x_{ij} + 1|), \tag{2.3}$$

– piece-wise linear sigmoid function with [0,1] output [6,8]:

$$f(x_{ij}) = \begin{cases} 0, & x_{ij} < 1 \\ x_{ij}, & 0 \leq x_{ij} \leq 1 \\ 1, & x_{ij} > 1 \end{cases}, \tag{2.4}$$

*Sigmoid function has the following properties $y_{ij} = f(x_{ij})$, i.e. $|f(x_{ij})| \leq c = const.$, and $(df(x_{ij})/dx_{ij}) \geq 0$.

– nonlinear function [7]:

$$f(x_{ij}) = \frac{2}{\pi} tan^{-1}(\frac{\pi}{2} K x_{ij}), \tag{2.5}$$

etc.

In [8] more general output function with its own dynamics is proposed:

$$\dot{y}_{ij} = -y_{ij} + f(x_{ij}), \tag{2.6}$$

as a higher order dinamical system.

The delay template elements contribute two additional terms:

$$\sum_{C(kl)\in N_r(ij)} A^{\tau}_{ij,kl} y_{kl}(t-\tau) + \sum_{C(kl)\in N_r(ij)} B^{\tau}_{ij,kl} u_{kl}(t-\tau). \tag{2.7}$$

In the case of single variable $\tilde{A}$ and $\tilde{B}$, the linear (space-invariant) cloning templates are represented by the following additive terms [1,2]:

$$\sum_{C(kl)\in N_r(ij)} A_{ij,kl} y_{kl}(t) + \sum_{C(kl)\in N_r(ij)} B_{ij,kl} u_{kl}(t). \tag{2.8}$$

In this case, when the template is space invariant, each cell is described by simple identical cloning templates defined by two real matrices A and B. Continuous input(output) signal values are presented by values in the range $[-1, 1]$ or $[0, 1]$.

Without loss of generality we can assume [1, 2]:

$$|u_{ij}(t)| \leq 1, \;\; |x_{ij}(0)| \leq 1. \tag{2.9}$$

Now in terms of definition 1.2 we can make a generalization of the above dynamical systems describing CNNs. For a general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^s_{ij}), \tag{2.10}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usualy a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I^s_{ij} &= A_{ij,kl} x_{i+k,j+l} + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ &\quad + \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \tag{2.11}$$

The first term $A_{ij,kl} x_{i+k,j+l}$ of (2.12) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

3. Bifurcation, periodic solutions and chaos in CNNs. CNNs are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets.

Consider a CNN described by the normalized system equations:

$$\frac{dx(\tau)}{d\tau} = -x(\tau) + Ay(\tau) + Bu + I = F(x). \tag{3.1}$$

The state vector x is produced by lining up every row of the cell states in sequence, then we have $x \in \mathbf{R}^n$, $n = M.M$ and the relation x_k to x_{ij} is given by [6,8,13]

$$x_k = x_{ij}/V_{sat}, \;\; i = ((k-1) div\, M) + 1, \;\; j = ((k-1) mod\, M) + 1,$$

V_{sat} is the saturation voltage of the cell. The vector u is the input of the network, I is the offset of cells, both are assumed to be constant, A and B are usually sparse matrices with a banded structure containing the template coefficients at proper places. The output $y \in \mathbf{R}^n$ is a piece-wise linear sigmoid function (2.3). For the analysis of the dynamical system (3.1), the stability properties of its equlibrium points should be investigated. An associated linear system in the sufficient small neighborhood of an equilibrium point $\bar{x}$ of (3.1) can be given by

$$\frac{dz}{dt} = DF(\bar{x})z, \tag{3.2}$$

where $z = x - \bar{x}$, and $DF(\bar{x}) = J$ is known as the Jacobian matrix of the equilibrium point and can be computed by

$$J_{ij} = \frac{\partial F_i}{\partial x_j}|_{x=\bar{x}}. \tag{3.3}$$

According to the stability theory of dynamical systems, it is well-known that if all eigenvalues of J have negative real parts, then the equilibrium point $x = \bar{x}$ is asymptotically stable. On the contrary, if one of the eigenvalues of J has a positive real part, then the equilibrium point $\bar{x}$ is unstable. In general, if none of the eigenvalues of J have zero real part, then the equilibrium point $\bar{x}$ is hyperbolic.

In [5] the dynamical system (3.1), is considered in the case when the output y is allowed to exibit hysteresis. In other words, the hysteresis output $y = h(x)$ is a real functional determined by an "upper" function h_U and a "lower" function h_L. (Fig.2)

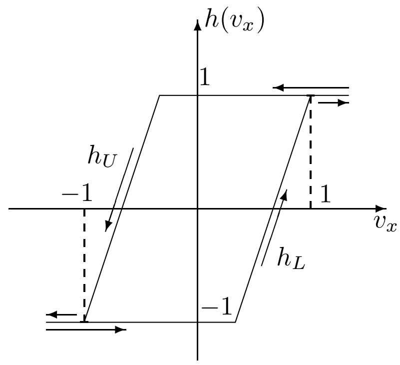

Fig. 2. Hysteresis nonlinearity

The functions h_U and h_L are real valued piece-wise continuous, differentiable functions. Moreover, $h(x)$ is odd in the sence that

$$h_U(x) = -h_L(x), \tag{3.4}$$

and also $h_U = h_L$ for $|x|$ sufficiently large.

Because of the hysteresis type of nonlinearity in each cell of a CNN, it is resonable to divide $M \times M$-dimensional Euclidean space into different types of regions. Stability region (SR) is defined for all x, such that $|x| > 1$ and $|h(x)| \geq 1$. In the hysteresis region (HR), $|h(x)| < 1$ for all $|x| \leq 1$. The definition of this region comes from hysteresis nonlinearity in the feedback system of our CNN (3.1). Partial stability region (PSR) is considered for $|x| \leq 1$, $|h(x)| > 1$ and $|x| > 1$, $|h(x)| = 1$. Then the following theorems hold:

Theorem 3.1. *In the stability region (SR) any state equilibrium point for our CNN will be asymptotically stable. In other words, after the transient has decayed to zero, any trajectory within this stability region will converge asymptotically to the corresponding unique state equilibrium point.*

Theorem 3.2. *In the hysteresis region (HR), for any $|x| \leq 1$, such that $|h(v_x)| \leq 1$, the state equilibrium points will be unstable. In other words, after the transient has decayed to zero, if $h_U(x) = -h_L(-x)$, any trajectory will not be convergent to the state equilibrium point.*

Theorem 3.3. *In the partial stability region we have two possible cases:*
i). $|x| > 1$, then the state equilibrium points are stable;
ii) $|x| \leq 1$, then the state equilibrium points are unstable.

Remark 3.1. From the above theorems one can conclude that a CNN with hysteresis can only converge to a state equilibrium point in the regions in which $|x| > 1$ (stability region and part of the partial stability region). If the network has reached a stable state its outputs can only be $+1$ or -1. In other words we have binary outputs, which property has a lot of applications in pattern recognition and signal processing.

In [13] the following two-cell autonomous CNN with opposite-sign template is considered:

$$\begin{aligned} \dot{x}_1 &= -x_1 + pf(x_1) - sf(x_2), \\ \dot{x}_2 &= -x_2 + sf(x_1) + pf(x_2), \end{aligned} \tag{3.5}$$

$p > 1$, $s > 0$, f is piece-wise linear function (2.3). In this system, Hopf-like bifurcation has been found, at which the flow arround the origin (equilibrium point of (3.5)) changes from asymptotically stable to periodic.

Let us rewrite (3.5) in the following form:

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, \mu) = -x_1 + (1+\mu)f(x_1) - sf(x_2), \\ \dot{x}_2 &= F_2(x_1, x_2, \mu) = -x_2 + sf(x_1) + (1+\mu)f(x_2), \end{aligned} \tag{3.6}$$

where assume that $p - 1 = \mu \in (-\delta, \delta)$, $\delta > 0$, $s > 0$.

Obviously (3.6) has always an equilibrium point at $x = 0$. The Jacobian matrix is given as

$$J_0 = DF(0, \mu) = \begin{bmatrix} \mu & -s \\ +s & \mu \end{bmatrix}.$$

The following result concerning the global bifurcation has been proved in [13]:

Theorem 3.4. *For the system (3.6)*

1). $\mu = \mu_1^* = s$ is a global bifurcation point of the system in which new equilibria are created;

2). for $\mu \in (\mu_2, \mu_1^*)$ with $0 < \mu_2 < \mu_1^*$ there is a stable limit cycle surrounding the origin,

3). for $\mu \in (\mu_1^*, \mu_3)$ with $\mu_3 > \mu_1^*$ although the origin is an unstable focus, the network is completely stable, almost all solutions tend to one of the stable equilbria created in this bifurcation.

In [6] an analogous two cell autonomous CNN has been considered where $f(x) = \frac{x^3}{3} - x$. Then the following theorem can be proved [6]:

Theorem 3.5. *For the two-cell CNN (3.6) with $f(x) = \dfrac{x^3}{3} - x$:*

i). $\mu = \mu^* = 0$ is a bifurcation point of the system. This is a local bifurcation of the only system equilibrium;

ii). for $\mu \in (-2, \mu^*)$ the origin is a stable focus;

iii). for $\mu \in (\mu_1, -2)$, some $\mu_1 < -2 < \mu^*$ the origin is an unstable focus surrounded by a stable limit cycle.

In [14] a chaotic attractor in a three-cell autonomous CNN has been reported. The dynamics of the system can be described by the set of ODEs:

$$\begin{aligned} \dot{x}_1 + x_1 &= p_1 f(x_1) - s f(x_2) - s f(x_3) \\ \dot{x}_2 + x_2 &= -s f(x_1) + p_2 f(x_2) - r f(x_3) \\ \dot{x}_3 + x_3 &= -s f(x_1) + r f(x_2) + p_3 f(x_3), \end{aligned} \tag{3.7}$$

where $p_1 > 1$, $p_2 > 1$, $p_3 \leq 1$, $r, s > 0$, the input u and the bias current I are set to zero. By solving (3.7) with the following parameter set: $p_1 = 1.25$, $p_2 = 1.1$, $p_3 = 1$, $s = 3.2$, $r = 4.4$ and initial condition $x(0) = (0.1, 0.1, 0.1)$, a strange attractor can be observed.

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КЛЕТЪЧНО НЕВРОННИ МРЕЖИ – ДИНАМИКА И КОМПЛЕКСНОСТ

Анжела Славова

В тази статия са представени Клетъчно Невронни Мрежи (КНМ). От тяхното откриване през 1988г., изследването им е насочено към много широк клас от задачи и явления. Много изследователи са допринесли за изучаването на КНМ като за целта са използвани различни математически методи. КНМ представлява аналогова динамична процесорна мрежа, изградена от клетки, които съдържат линейни кондензатори, линейни резистори, линейни и нелинейни контролни източници. В тази статия се прави обзор на основните типове динамични уравнения описващи КНМ. Изучават се динамичните свойства на КНМ с хистерезис във веригата за обратна връзка. За някои класове КНМ са доказани бифуркации, периодични решения и хаос.