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BERNOULLI TRIALS: EXTENSIONS, RELATED PROBABILITY DISTRIBUTIONS AND MODELING POWERS*

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Here we follow a pattern of constantly evolving scheme of the classical series of Bernoulli trials. Related random variables, probability distributions, processes, models, results and limit theorems are briefly marked. Their areas of application are selectively presented. The ways of this development, some new ideas, new results and directions of further studies and applications are the main goal of this article.

1. Introduction and historical remarks. *Bernoulli trials* are repeated trials that obey the following conditions:

- Each trial yields one of two outcomes, often called "success" (S), and "failure" (F);
- For each trial the probability of S is the same, usually denoted by $p = P(S)$, and the probability of F is denoted by $q = 1 - p = P(F)$;
- The trials are independent: the probability of S in a trial does not change, given any information about outcomes of other trials.

Tossing a coin, rolling a die, random sampling of a ball from an urn of equal balls with replacements, are the usual examples of sequences of Bernoulli trials.

Let ω be the random outcome of a Bernoulli trial (BT). Define variables X_i associated with the $i-$th BT, and $X_i(\omega) = 1$ if $\omega = S$, and $X_i(\omega) = 0$ if $\omega = F$. The quantity $p = P(X_i = 1)$ is called *parameter* of the *Bernoulli distribution*, represented by the table

$$\begin{array}{c|ccc} X_i & 0 & , & 1 \\ \hline p(x) & 1-p & , & p \end{array} \; ,$$

and the random variables (r.v.) X_i are called Bernoulli' r.v.

Denote by

$$S_n = X_1 + X_2 + \cdots + X_n \tag{1}$$

the total number of S-s in n consecutive BT-s. The ratio $\dfrac{S_n}{n}$ is called *relative frequency*. The first significant result in probability is considered the following:

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Bernoulli Theorem (Law of Large Numbers): If $P(S) = p$, and if $\varepsilon > 0$, then

$$P\left\{|\frac{S_n}{n} - p| < \varepsilon\right\} \to 1 \quad \text{as} \quad n \to \infty.$$

In other ways of writing, in textbooks one can see this result in the form $S_n/n \xrightarrow{p} P(S)$, meaning that the relative frequency of successes approaches (in probability) the probability of success in a single trial, as the number of trials increases indefinitely.

The remarkable behavior of frequency data was first observed in the fields of games of chance. At the early epoch, it was observed that in all current games with cards, dice, etc. the frequency of a given result of a certain game seem to cluster in the neighbourhood of some definite value, when the game was repeated large number of times. Origins (about 1650) and first development of mathematical theory of probability we due to the hands of Pascal, Fermat, Huygens, and Jacob Bernoulli. The same type of regularity was found later to occur with various demographic data (Laplace, Galton, Maltus), and the theory of population statistics was based on this fact (Bernstein. Pearson, Fisher, Kramer, and many others). The long-run stability of frequency ratios is a general characteristic of random experiments performed under uniform conditions.

The idea of an infinite parent population (in statistics) is a mathematical fiction (abstraction) of the same kind as the idea that a given experiment may be repeated an infinite number of times (as in these infinite Bernoulli trials). Similar is the idea of sampling from an infinite population. And even though, in modeling one should accept this idea and make the necessary assumptions in order to get some reasonable model to describe the uncertainty even in a single item, in an unique uncertain process. We think like having infinitely many copies of the uncertain situation, and one of it may occur. Whenever we base a result on this assumption, we will refer to it as on *The Infinite Population Presumption*, (briefly *TIPP*). Notice, that some new areas in the science such as Utility Theory, Decision Making, Games Theory, and others, are based almost completely on TIPP.

2. Classical development. With any sequence of BT the following r.v.'s and probability distributions are usually associated (any textbook on probability should have it):

2.1. Counting variables. The Geometric Distribution. This is the number $\nu = \min\{n; X_1 + \cdots + X_n = 1\}$, where the first success occurs. It is well known that

$$P(\nu = k) = pq^{k-1}, \quad \text{with } q = 1 - p \quad k = 1, 2, \dots ,$$

and p is called the parameter of the Geometric distribution. Notation $\nu \sim Ge(p)$. This variable is useful to describe counts of trials until a first success occurs, and is a frequently used component of numerous models of random processes of mixed discrete/continuous type.

The Negative Binomial Distribution. This is the number $\nu_r = \min\{n; X_1 + \cdots + X_n = r\}$, where the first r successes occur, and r is a positive integer. It is well known that

$$P(\nu_r = r + k) = \binom{-r}{k} p^r (-q)^k, \quad \text{with } q = 1 - p \quad k = 0, 1, 2, \dots ,$$

and (r, p) are called then parameters of the Negative Binomial distribution. Notation

$\nu_r \sim NB(r,p)$. These variables are used to describe the waiting time until some fixed number of events will occur. Also component of more complex models, e.g. in reliability, demographic models, insurance. Notice that $NB(1,p) = Ge(p)$.

The Binomial Distribution. This is the distribution of the r.v. S_n given by equation(1). It is well known that

$$P(S_n = k) = \binom{n}{k} p^k q^{n-k}, \quad \text{with } q = 1-p \quad k = 0, 1, \ldots n,$$

and the pair (n,p) is called the parameter of the Binomial distribution. Notation $S_n \sim Bin(n,p)$. One of the most used in statistical modeling (e.g. in the study of ordered statistics), in reliability (e.g. k-out-of-n systems), in insurance and risk models (e.g. number of claims within a portfolio of uniform subscribers), and many other situations.

A remarkable relationship. The waiting times ν_r until the occurrence of the $r-$th success, and the number of successes S_n within the first n BT are related by the equations

$$P(S_n \geq r) = P(\nu_r \leq n), \quad \text{for any} \quad n \geq r, \tag{2}$$

which have a nice and obvious interpretations in terms of events. It also allows to obtain some limit approximations from either of the two random variables (e.g. Poisson, or normal approximations for the binomial distributions may be successfully used and for the negative binomial distribution).

2.2. Generalizations based on relaxing conditions on BT. First generalizations come from relaxing one, or more of the three conditions which specify the Bernoulli trials. Some of these are less known.

Poisson models: If one drops the second requirement for same probability of success in each trial, and allows, say, $P(X_i = 1) = p_i$ for $i = 1, 2, \ldots$, then one arrives to first generalizations of the Bernoulli sequence of trials called Poisson trials. All the above introduced r.v.'s stay defined as before, and their distributions now will carry an attachment "Poisson". Hence, we get Geometric-Poisson, Binomial-Poisson, and Negative Binomial-Poisson distributions. We skip details. Just notice that the Poisson schemes appear as a convenient tools to model dynamic situations where the subscript i is interpreted as a time unit, or where the conditions of experiment change with the number of already completed trials (e.g. sampling from finite populations, or changing the urn in sequence of sampling, or doing experiments within different seasons, or at different places).

Dependent trials: Another kind of generalizations is obtained after dropping the requirement for independent trials in the Bernoulli sequence. Possible dependence between $n-$th and $(n+1)-$st trials may be described as a *Markov chain* by the requirements

$$\begin{array}{lll} P(X_{n+1} = 0 | X_n = 0) = p_{00}; & P(X_{n+1} = 1 | X_n = 0) = p_{01}, & p_{00} + p_{01} = 1 \\ P(X_{n+1} = 0 | X_n = 1) = p_{10}; & P(X_{n+1} = 1 | X_n = 1) = p_{11}, & p_{10} + p_{11} = 1. \end{array} \tag{3}$$

An initial distribution $P(X_0 = 0) = 1 - \alpha$, and $P(X_0 = 1) = \alpha$ should be assumed. The dependent Bernoulli trials are then modeled with a special choice of the transition probabilities $P_{ij},\ i,j = 0,1$ which satisfy that the unconditional probabilities are

$$P(X_n = 0) = 1 - p, \quad \text{and} \quad P(X_n = 1) = p,\ i = 0, 1, 2, \ldots\ , \tag{4}$$

and $\{X_n\}_{n=0}^{\infty}$ are still the r.v.'s reflecting the outcomes S, or F in the $n-$th trial.

These models are useful in studies related to molecular biology, reliability, online quality control. However, there is some complexity in understanding this Markov chain (MC) approach, and it can be overcome by an alternative approach, which we call *Correlated Bernoulli trials*, briefly (CBT).

In the CBT the correlated structure between the trials is expressed by the requirement

$$Corr(X_n, X_{n+1}) = \rho, \quad \rho \in [-1, 1], \quad n = 0, 1, 2, \dots, \tag{5}$$

where the quantity ρ describes possible dependence between two adjacent trials. It can be easily shown that the correlated sequence $\{X_n\}_{n=0}^{\infty}$ forms a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} q + \rho p & p(1-\rho) \\ q(1-\rho) & p + \rho q \end{bmatrix}. \tag{6}$$

A comparison between (3), (4), and (6) will show that the CBT and the Markovian model at the beginning are equivalent forms of presenting dependent BT. Independent BT are presented by the value $\rho = 0$. In case of CBT the dependence is vanishing with the distance between trials. It is true that

$$Corr(X_{n+m}, X_m) = \rho^n, \quad n = 1, 2, \dots,$$

for any fixed m. As a matter of fact, Edwards (1960) uses this MC approach to study the number S_n of boys in a family of n children, assuming a positive correlation ρ between the gender of two consecutively born babes. He found that the p.g.f. of S_n is given by the equation

$$G_{S_n}(z) = \mathbf{E}(Z^{S_n}) - [pz, q] \begin{bmatrix} (p + \rho q)z & q(1-\rho) \\ p(1-\rho)z & q + \rho p \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{7}$$

The random variables ν, S_n, and ν_r associated before with the BT and the $Ge(p)$, $Bin(n, p)$, and $NB(r, p)$ distributions are now defined in the same way as in Subsection 2.1, and carry the prefix *correlated* distributions. Now they have an additional parameter, and this is ρ. In this way we get:

Correlated Geometric distribution $\nu \sim CGe(p; \rho)$: It is presented by the probability generating function (p.g.f.)

$$P(z) = \mathbf{E}(\nu^z) = p\frac{1 - \rho z}{1 - (q + \rho p)z}, \quad |z| \le 1. \tag{8}$$

Correlated Negative Binomial distribution $\nu_r \sim CNB(r, p; \rho)$: It is presented by the p.g.f. which is the r^{th} power of the m.g.f. of the $CGe(\rho, p)$ given in (8). We don't write it explicitly. Notice that from the p.g.f. one can get all the interesting average characteristics of the respective r.v. as values of its derivatives at $z = 1$, and use them purposely. The p.g.f.'s are also used to study the limit behavior of the respective r.v.'s when one, or more parameters tend to some limiting values (the weak convergence).

Correlated Binomial distribution $S_n \sim CBin(n, p, \rho)$: It is presented by the p.g.f.

$$P_{CS_n}(z) = \mathbf{E}(z^{S_n}) = (1 - \rho)(1 - p + pz)^n + \rho(1 - p + pz^n), \quad |z| \le 1. \tag{9}$$

Notice, that when $\rho = 0$ the above p.g.f. turns into the p.d.f. of the usual $Bin(n, p)$. This result is a starting point to introduce a series of *correlated versions* of classic continuous distributions (see subsection 2.5).

Remark. Correlated binomial distribution with parameters n, p and ρ is introduced implicitly by Tallis (1962), and rediscovered and studied later by Luceño (1995), and Luceño and Caballos (1995). It is due to Kolev (1999) to notice that correlated distributions coincide with the so called *zero inflated discrete distributions* of the same type.

A discrete distribution which is modified by increasing the probability for one value (say, it is $X = k_0$) and the remaining probabilities being multiplied by an approximate constant to keep the sum of probabilities equal to 1, is called *inflated distribution.* By denoting $\tilde{X}$ the "inflated" original variable X, this modification is expressed by the relations

$$P(\tilde{X} = k_0) = 1 - \alpha + \alpha P(X = k_0); \quad P(\tilde{X} = k) = \alpha P(X = k),\ k \neq k_0.$$

In most applications X has as a support the non-negative numbers, and $k_0 = 0$. Then the inflated $\tilde{X}$ has support "with added zeros", and $\mathbf{E}(\tilde{X}^r) = \alpha \mathbf{E}(X^r)$. If $\alpha > 1$, we have "deflated distribution". The greatest possible value for α is $1/[1 - P(X = k_0)]$. Kolev (1999) noticed that the correlated distributions mentioned here are the same as the zero inflated distributions with $\alpha = 1 - \rho$. Thus, when $\rho > 0$ the correlated r.v's behave as inflated, and when $\rho < 1$ then they behave like deflated variables. Zero inflated distributions are appropriate alternatives for modeling clustered samples. For example, $\tilde{X}$ is suitable for modeling the distribution of samples drown from populations, which consists of two subpopulations, one containing mostly zeros, while in the other, any integer value may be observed.

2.3. Combined Bernoulli trials. Runs and related distributions. In a sequence of BT a new success can be defined as a series of certain results in k consecutive trials, called *run of k outcomes.* Usually series of consecutive successes, or failures of given length are considered. As soon as such a sequence is obtained, it is said that the respective quota of successes (or failures) is attained. However, the interests of genetic engineering may require other runs, not studied yet. Studies started less than 20 years ago and quickly attract attention (Johnson et al. 1992). Let $\nu(k)$ be the number of trials when the first *run of k successes* is completed. Then the distribution of the r.v. $\nu(k)$ is called **Geometric distribution of order k.** It is determined by the p.g.f.

$$G_{\nu(k)}(z) = \frac{p^k z^k (1 - pz)}{1 - z + qp^k z^{k+1}}. \tag{10}$$

This is Philippou et al. (1983) geometric distribution of order k. The case of $k = 1$ gives the ordinary $Ge(p)$.

The waiting time of r successes runs of length k has a **Negative Binomial distribution of order k.** This is a $r-$fold convolution of geometric distribution of order k, and its p.g.f. is the r^{th} power of $G_{\nu(k)}(z)$ from (10).

The number of occurrences of k consecutive successes in n independent BT has **Binomial distribution of order k.** This distribution is important in the theory of $k-$out-of$-n$ reliability systems. It has been studied by Hirano et al. (1991). The respective r.v. $X-$distribution is given by the equations

$$P(X = m) = p^n \sum_{i=0}^{k-1} \sum_{i_1,\ldots,i_k,m} \binom{i_1 + \cdots + i_k + m}{i_1, \ldots, i_k, m} \left(\frac{q}{p}\right)^{i_1+\cdots+i_k},$$

where the summation is on all non-negative integers $i_1, \ldots, i_k$ which satisfy the relations $i_1 + 2i_2 + \cdots + ki_k = n + i - mk$, and $\binom{n}{i_1, \ldots, i_k} = \frac{n!}{i_1! \ldots i_k!}$.

Since Philippou many authors contributed to the theory of distributions of order k. Aki (1985) considered the first occurrence of a series of k successes in dependent (similar to correlated) BT which form a MC. Balasubramanian et al. (1993) analyze, say, the waiting times when either k consecutive successes, or l consecutive failures will be completed, or times when both will be completed, in a sequence of dependent BT (i.e. on a two-state MC). In two papers Kolev and Minkova (1999) generalize this study to an arbitrary MC, with a specially defined states of success, and failures. A good view on the related problems and possible applications can be found in Fu and Kuotras (1996). Kolev illustrates applications of distributions of order k in a Technical Report (1999).

Our opinion is that here are lots of opportunities for other, clear mathematical studies on number of occurrences of a well defined sequence of symbols within a given number of trials, on competing sequences (who will occur first, who later), in the frame of independent, or dependent BT, and other generalizations of discrete distributions, say, of mixed order (k, l), etc. will appear. Their application in the studies of genetic biology seem obvious.

2.4. More than one type of success. Multinomial distributions. Here we just notice that the BT have a natural extension to a case where more than two types of outcomes in a single trial may occur (compare the outcomes in tossing a coin, and rolling a die). Let $\omega_1, \ldots, \omega_m$ be the outcomes in a single trial, and $p_1, \ldots, p_m$ be their probabilities respectively. If denote by $X_1, \ldots, X_m$ the number of occurrences of these outcomes in n consecutive trials, then the joint distribution of $X_1, \ldots, X_m$. This is the well known **Multinomial distribution**, whose particular case for $m = 2$ is the Binomial distribution. Models of dependent trials, at the best of our knowledge, are note well explored, neither the distributions of fixed portions of outcomes. However, such models also may be important for molecular biology and genetic engineering.

2.5. Limit theorems and related distributions. It is well known that when the parameters of a distribution go close to their boundary values, the respective r.v. starts behave strange (e.g. when $p \to 1, R$ then $\nu \to \infty$ in the $Ge(p)$ model), and practice better use approximations than the original distributions. Here we mention three classical approximations which make a bridge between distributions, and also illustrate how new classes of distributions have been introduced. Mathematically all of these facts mean convergence in distribution, and can be analytically proven by showing that the p.g.f. of the right-hand side goes, under the required conditions, to the p.g.f. of the left-hand side (an application of the *continuity theorem* and the *integral transforms* of functions).

The Exponential distribution $Exp(\lambda)$ as a limit of $Ge(p)$. It is known that

$$\lim_{p \to 0} P\left(\frac{\nu}{\mathbf{E}(\nu)} \le x\right) = 1 - e^{-x}, \quad x \ge 0. \tag{11}$$

In words it means that the r.v. ν asymptotically behaves as an $Exp(\lambda = 1/p)$ distributed r.v. when p approaches 0. Such fact is symbolically indicated as

$$Ge(p) \;\approx\; Exp(1/p) \quad as \quad p \to 0, \tag{12}$$

and is used to approximate the respective probabilities when direct calculations are complicated.

The **Correlated Exponential distribution** $CExp(\lambda;\rho)$ is obtained in the same way as $Exp(1)$ in (11), and the fact is

$$CGe(p) \approx CExp(1/p;\rho) \quad as \quad p \to 0,$$

where $CExp(\lambda;\rho)$ is defined by the p.d.f. $f_{CExp}(\lambda,\rho;x) = 1-\rho+\rho\, e^{-\lambda x}$.

The Poisson distribution $Po(\lambda)$ can be obtained as an approximation

$$(13) \qquad Bin(n,p) \approx Po(\lambda = np) \quad as \quad n\to\infty,\ p\to 0\ and\ np\to\lambda.$$

Following the same classical way Luceño (1995) derived the **Correlated Poisson distribution** $CPo(\lambda;\rho)$ from correlated negative binomial distribution, whose p.g.f. is $P_{CPo}(t) = \rho + (1-\rho)e^{\lambda(t-1)}$.

Analogously, following the way of deriving **The Normal approximation** $N(\mu = np, \sigma = \sqrt{npq}) \approx Bin(n,p)$ **as** $n\to\infty, pq\neq 0$ **to the Binomial distribution** from $\lim_{n\to\infty} P\left(\frac{S_n - np}{\sqrt{npq}} \le x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\, du$, one can introduce the **Correlated Normal distribution** as a respective approximation to the $CBin(n,p;\rho)$ from (7).

Other similar generalizations with a starting point from distributions of order k, or from multinomial correlated distributions are also possible. We anticipate lots of joy for whom find the proofs, and respective explicit forms of such approximations.

3. Trials extended in time. Environmental and risk modeling. Now let us suppose that it takes some considerable time to perform any particular trial in the Bernoulli sequence. It is clear that the trial ends with no success only when it is completed. However, if there is a success it is immediately discovered. For instance, consider the car accident of a driver: just by the end of the year it is made clear if no accident happened, and an accident is immediately registered. Let each trial requires some fixed time $c>0$, and let Y_i be the elapsed time from the beginning of the trial up to the moment when S occurs, presumably it occurs within this trial. Let X be the total time elapsed from the start of a sequence of BT until S occurs. Then we should name the distribution of the r.v. X **Extended in time Geometric distribution Briefly** $ETCe(p,Y,c)$**.** If assume the sequence $\{Y_i\}_{i=1}^{\infty}$ independent, identically distributed, then X has a continuous distribution when Y_i are continuous on $[0,c]$. Dimitrov and Khalil (1992) found the form of this distribution

$$(14) \qquad F_X(x) = 1 - p^{[\frac{x}{c}]}\left\{1-(1-p)F_Y\left(x - \left[\frac{x}{c}\right]c\right)\right\}, \quad x\ge 0,$$

where $F_Y(y)$ is the distribution of Y_i's, and $[\frac{x}{c}]$ is the integer part of the number x/c. They also discuss the use of such distributions in modeling environmental processes with periodic behavior. And here is the key to this modeling: environmental processes are embedded into conditions that are periodically repeated, due to the seasonal character of our surrounding (refer to the above example with the car accidents). Most of the events in insurance business have this subordination with the environmental conditions, which act as conditions of a slowed down BT, and are periodically repeated. Here we see the main stream of applications of these distributions and models, besides of the truly mathematical and probabilistic problems and solutions. Dimitrov (1998) notice also possible applications in technical studies.

Dimitrov and Chukova (1992) discovered the *Almost-lack-of-memory property* in random variables with the distribution (14) in relation with the blocking time in service on non-reliable server, and called it $ALM(p, F_Y, c)$ distribution, because of the ALM property. Later Dimitrov et al. (1997) fully analyzed the properties of the ALM distributions by using the idea of Kotz and Shanbhag (1980) of use of failure rates as more natural physical characteristic in describing probability behavior of r.v.'s. It has been found that the ALM-, or $ETGe$-distributions have a periodic failure rate. This perfectly matches the above mentioned concept of $TIPP$ for BT embedded in periodic random environment, and pays of in environmental, risk, and insurance modeling.

Chukova et al. (1993), and Dimitrov et al. (2000) discuss the use of multiple superposition of r.v.'s with distribution of X on the interval axis, and found the number of successes $N_{t,t+s)}$ on any time interval $[t, t+s)$ to have a Poisson distribution with periodic intensity $\lambda(t) = f_X(t)/[1 - F_X(t)]$, where $F_X(x)$ is defined by (14). This fact let them to find the representation

$$N_{[t,t+s)} = M_1 + \cdots + M_{[\frac{s}{c}]} + N_{s-[\frac{s}{c}]c},$$

where $\{M_n\}_{n\geq 1}$ are i.i.d. Poisson r.v.'s of parameter $p = \int_0^c \lambda_X(t)\, dt$, and are independent with the last component of the above sum. The result works in the studies of risk processes where the accumulated losses within an interval $[t, t+s)$ are presented by the sum $C_1 + \cdots + C_{N_{[t,t+s)}}$ called *compound process*. We do not go into details, just notice that same results help to improve some known evaluations of probability of ruin in risk analysis, where compound processes are key component (ref. Bowers et al. 1986, Gerber 1979).

By the best of our knowledge, nothing has been done to establish any analog of either the **Extended in time Negative Binomial distribution**)briefly $ETNB(r, p, Y, c)$), or the **Extended in time Binomial distribution ($ETBin(p, Y, c)$)**. No models alike the *order k distributions* are studied. Very few results for approximations of the ALM distributions. Even the classic Bernoulli low of large numbers should look differently because of dependence on time parameter. Notice that such models may be difficult because of dependence of the intensity of successes on the time of occurrence. However, the **Extended in time Poisson trials** ($ETPT$) found a good development and area of applications. Dimitrov and Chukova (1999) constructed a model of $ETPT$, and found the form of the distribution of the time X to first occurrence of success:

$$F_X(x) = \sum_{j=1}^{k} p_j \prod_{i=1}^{j-1}(1 - p_i) + \left\{ p_{k+1} \prod_{i=1}^{k}(1 - p_i) \right\} F_{Y_{k+1}} \left(x - \left[\frac{x}{c} \right] c \right),$$

and in case of all $p_i = p$ this result coincides with (14). An application to *tornado-watch modeling* illustrates the particular case of periodic sequence of probabilities for success $\{p_i\}_{i=1}$ which describes so called *driving periodic conditions*. Similar approach has been demonstrated earlier in Dimitrov et al. (1998) to model a dynamic population growth in periodic environment, where the *multiplicative almost-lack-of-memory* in consecutive trials is introduced. The last concept is not well explored yet, and has no analogous in the classical schemes listed above.

Extended in time correlated BTs are on the way of exploration. A need of such type of models is obvious. There is an intuitively clear correlation between types of the seasons in the environmental surrounding (warm-warm, severe-severe, dry-wet, etc.). This means that extended in time BT should be considered correlated. The specific noticed in Section 2.2 remains, and new components appear. These are the times of performing trials of different kind. Obtained results give us a hope to report them soon. The study is still in progress.

4. Conclusions. Here we selected a specific development of an idea by abandoning original conditions, or by exploring it in deep with more focus and details. Any time we discover that as a result new mathematical objects appear (random variables, processes, probability distributions, properties) and then live their separate life as objects of further study, or subject to applications. In this way we notice some fields of hot studies, and blanc spots for exploration, all related to the oldest components on which the probability theory is build.

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БЕРНУЛЕВИ ОПИТИ. РАЗШИРЕНИЯ, ПОРОДЕНИ ВЕРОЯТНОСТНИ РАЗПРЕДЕЛЕНИЯ И МОЩ НА МОДЕЛИРАНЕ

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В статията е следван модела на постоянно еволюираща схема на класическата поредица от Бернулеви опити. Накратко са отбелязани свързани с тях случайни величини, вероятностни разпределения, процеси, модели, резултати и гранични теореми. Представени са селективно техните области на приложение. Главната цел на тази работа са пътищата на развитие и обобщение, някои нови идеи и приложения.