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$\boldsymbol{R}$-CORRESPONDING NETS IN $\boldsymbol{V_n}$

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Using the tensor of Richi R-corresponding nets are introduced. With the help of the found relations between the coefficients of the derivative formulae, invariant characteristics of orthogonal and equidistant nets are obtained.

1. Let the Richi tensor $\mathbf{R}_{is} \neq 0$ in a Riemannian space $\mathbf{V}_n$. With the help of the mutual tensor g^{is} of metric one g_{is} we introduce

(1) $$\mathbf{R} = \mathbf{R}_{ik}\, g^{ks}.$$

The net $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ is defined by the independent unit fields of vectors $\underset{\alpha}{v}^i$ $(\alpha = 1, \dots, n)$ in the space $\mathbf{V}_n$.

We determine the net $(\underset{1}{w}, \underset{2}{w}, \dots, \underset{n}{w}) \in \mathbf{V}_n$ with the independent fields of directions

(2) $$\underset{\alpha}{w}^i = R^i_s \underset{\alpha}{v}^s$$

Definition 1. *The nets* $(\underset{1}{v}, \underset{2}{v}, \dots, \underset{n}{v})$ *and* $(\underset{1}{w}, \underset{2}{w}, \dots, \underset{n}{w})$ *will be called R-corresponding.*

Let the derivative formulae are:

(3) $$\nabla_k \underset{\alpha}{v}^i = \overset{\sigma}{\underset{\alpha}{T}}_k \underset{\sigma}{v}^i, \quad \nabla_k \underset{\alpha}{w}^i = \overset{\sigma}{\underset{\alpha}{P}}_k \underset{\sigma}{v}^i.$$

From (2) and (3), we obtain:

$$\nabla_k\ \underset{\alpha}{w}^i = \overset{\sigma}{\underset{\alpha}{P}}_k \underset{\alpha}{w}^i = \nabla_k \left(R^i_s\ \underset{\alpha}{v}^s \right) = \nabla_k\ R^i_s\ \underset{\alpha}{v}^s + R^i_s \nabla_k\ \underset{\alpha}{v}^s = \nabla_k\ R^i_s\ \underset{\alpha}{v}^s + R^i_s \overset{\sigma}{\underset{\alpha}{T}}\ \underset{\sigma}{v}^s.$$

Thus we find:

(4) $$\overset{\sigma}{\underset{\alpha}{P}}{}_k\ \overset{i}{\underset{\sigma}{w}} = \nabla_k R^i_s\ \underset{\alpha}{v}^s + R^i_s \overset{\sigma}{\underset{\alpha}{T}}\ \underset{\sigma}{v}^s.$$

Taking into account (2) and (4) we obtain:

$$\overset{\sigma}{\underset{\alpha}{P}}\ R^i_s \underset{\alpha}{v}^s = \nabla_k\ R^i_s\ \underset{\alpha}{v}^s + R^s_i\ \overset{\sigma}{\underset{\alpha}{T}}\ \underset{\sigma}{v}^s,$$

from here we find

(5) $$\left[\left(\overset{\sigma}{\underset{\alpha}{P}}{}_k - \overset{\sigma}{\underset{\alpha}{T}}_k \right) R^i_s - \nabla R^i_s \delta^\sigma_\alpha \right] \underset{\sigma}{v}^s = 0.$$

As $\underset{\alpha}{v}^s$ are independent, from (5) it follows

$$\left(\overset{\sigma}{\underset{\alpha}{P}}_k - \overset{\sigma}{\underset{\alpha}{T}}_k \right) R^i_s - \nabla R^i_s \delta^\sigma_\alpha = 0.$$

Thus we prove:

Theorem 1. *The coefficients of the derivative formulae (3) for R-corresponding nets* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{n}{v})$ *and* $(\underset{1}{w},\underset{2}{w},\ldots,\underset{n}{w})$ *satisfy*

$$\underset{\alpha}{\overset{\sigma}{P}}_k = \underset{\alpha}{\overset{\sigma}{T}}_k, \quad \alpha \neq \sigma; \qquad \left(\underset{\alpha}{\overset{\sigma}{P}}_k - \underset{\alpha}{\overset{\sigma}{T}}_k\right) R^i_s = n\nabla_k R^i_s.$$

Corollary 1. *If the net* $(\underset{1}{v},\underset{2}{v},\ldots,\underset{n}{v})$ *and the net* $(\underset{1}{w},\underset{2}{w},\ldots,\underset{n}{w})$ *are orthogonal then the Richi tensor of the space* $\mathbf{V}_n$ *is co-variant constant.*

Proof. Since the $(\underset{1}{v},\underset{2}{v},\ldots,\underset{n}{v})$ net and $(\underset{1}{w},\underset{2}{w},\ldots,\underset{n}{w})$ net are orthogonal then according to [1] we have $\underset{(\alpha)}{\overset{(\alpha)}{P}}_k = \underset{(\alpha)}{\overset{(\alpha)}{T}}_k = 0$. That means that $\nabla_k R^i_s = 0$ and using (1) we obtain $\nabla_k R_{is} = 0$.

Corollary 2. *The field* $\underset{\alpha}{v}^i$ *(respectively* $\underset{\alpha}{w}^i$*) is parallely translated along the lines* $(\underset{\alpha}{v})$ *(respectively* $(\underset{\alpha}{w})$*) if and only if the field* $\underset{\alpha}{w}^i$ *(respectively* $\underset{\alpha}{v}^i$*) is parallely translated along the lines* $(\underset{\beta}{v})$ *(respectively* $(\underset{\beta}{w})$*).*

Proof. It follows from the fact [1] that the fields $\underset{\alpha}{v}^i$ and $\underset{\alpha}{w}^i$ are parallely translated along the lines $(\underset{\beta}{v})$ if and only if the conditions (6) $\underset{\alpha}{\overset{\sigma}{T}}_k \underset{\beta}{v}^k = 0$, $\underset{\alpha}{\overset{\sigma}{P}}_k \underset{\beta}{v}^k = 0$ are fulfilled.

Corollary 3. *The field* $\underset{\alpha}{v}^i$ *(respectively* $\underset{\alpha}{w}^i$*) is geodesic if and only if the field* $\underset{\alpha}{w}^i$ *(respectively* $\underset{\alpha}{v}^i$*) is parallely translated along the lines* $(\underset{\alpha}{v})$ *(respectively* $\underset{\alpha}{w}$*).*

Proof. Really, from $\underset{\alpha}{\overset{\sigma}{T}}_k \underset{\sigma}{v}^k = 0$ and (6) we obtain the statement.

2. Let Γ^k_{is} are the coefficients of the connection of the space $\mathbf{V}_n$. Let them be determined by the metric tensor g_{is}. Then:

$$\Gamma^k_{is} = \frac{1}{2} g^{km}(\partial_i g_{ms} + \partial_s g_{im} - \partial_m g_{is}).$$

In the space $\mathbf{V}_n$ introduce the connection

$$G^k_{is} = \frac{1}{2} R^{km}\,(\partial_i.R_{ms} + \partial_s R_{im} - \partial_m R_{is})$$

where the tensor R^{km} is the mutual of the Richi tensor R_{is}. Denote the co-variant derivative in the connection G by $'\nabla$. Then:

$$'\nabla_k R_{is} = 0. \tag{9}$$

Denote the Riemannian space $\mathbf{V}_n$ with a metric tensor $\mathbf{R}_{is}$ by $\overline{V}_n$.

Proposition 1. *Vector fields* $\underset{\alpha}{v}^i$ *and* $\underset{\alpha}{w}^i$ $(\alpha \neq \beta)$ *are orthogonal in* $\mathbf{V}_n$ *if and only if the net is orthogonal in the* $\overline{V}_n$.

Proof. From (1) follow the equations:

$$g_{is}\underset{\alpha}{v}^i\underset{\beta}{v}^s = g_{is}\underset{\alpha}{v}^i R^s_m \underset{\beta}{v} = g_{is}\underset{\alpha}{v}^i g^{sp} R_{pm}\underset{\beta}{v}^m = \delta^p_i R_{pm}\underset{\alpha}{v}^i\underset{\beta}{v}^m = R_{im}\underset{\alpha}{v}^i\underset{\beta}{v}^m, \qquad \alpha \neq \beta$$

or

$$g_{is}\underset{\alpha}{v}^i\underset{\beta}{v}^s = R_{is}\underset{\alpha}{v}^i\underset{\beta}{v}^s, \qquad \alpha \neq \beta. \tag{10}$$

Proof. The field $\underset{\alpha}{v}^i$ in the space V_n is orthogonal of the field $\underset{\alpha}{w}^i$ $(\alpha \neq \beta)$ $\Longleftrightarrow$ $g_{is}\underset{\alpha}{v}^i\underset{\beta}{w}^s = 0$. The net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ is orthogonal in the $\overline{V}_n$ $\Longleftrightarrow$ $R_{is}\underset{\alpha}{v}^i\underset{\beta}{v}^s = 0$, $\alpha \neq \beta$. Thus from (10) the truth of the proposition follows.

3. Introduce the co-vectors:

$$\underset{\alpha}{v}_s = g_{is}\underset{\alpha}{v}^i, \quad \underset{\alpha}{w}_s = g_{is}\underset{\alpha}{w}^i. \tag{11}$$

We shall prove:

$$\nabla_k \underset{\alpha}{v}_s = \overset{\sigma}{\underset{\alpha}{T}}_k\, \underset{\sigma}{v}_s, \qquad \nabla k \underset{\alpha}{w}_s = \overset{\sigma}{\underset{\alpha}{P}}_k\, \underset{\alpha}{w}_s. \tag{12}$$

Let $\nabla_k \underset{\alpha}{v}_s = \overset{\sigma}{\underset{\alpha}{Q}}_k\, \underset{\sigma}{v}_s$. From (3) and (11) we obtain $\nabla_k \left(g_{is}\underset{\alpha}{v}^i\right) = g_{is}\, \overset{\sigma}{\underset{\alpha}{T}}_k\, \underset{\sigma}{v}^i = \overset{\sigma}{\underset{\alpha}{T}}_k\, \underset{\sigma}{v}_s$, or

$$\overset{\sigma}{\underset{\alpha}{Q}}_k = \overset{\sigma}{\underset{\alpha}{T}}_k\,.$$

From (1), (2) and (11) it follows:

$$R^i_k\underset{\alpha}{v}^k g_{is} = g^{ip}R_{kp}\underset{\alpha}{v}^k g_{is} = \delta^p_s R_{kp}\underset{\alpha}{v}^k = R_{ks}\underset{\alpha}{v}^k,$$

or

$$\underset{\alpha}{w}_s = R_{ks}\underset{\alpha}{v}^k.$$

Let $\nabla_i \underset{\alpha}{w}_s = \overset{\sigma}{\underset{\alpha}{S}}_i\, \underset{\sigma}{w}_s$. From (1), (2), (3) and (11) we find:

$$\nabla_k \left(g_{is}\underset{\alpha}{w}^i\right) = \overset{\sigma}{\underset{\alpha}{P}}_k\, g_{is}\underset{\sigma}{w}^i = \overset{\sigma}{\underset{\alpha}{P}}_k\, g_{is}R^i_m\underset{\sigma}{v}^m =$$

$$= \nabla_k \left(g_{is}\underset{\alpha}{w}^i\right) = \overset{\sigma}{\underset{\alpha}{P}}_k\, g_{is}\underset{\sigma}{w}^i = \overset{\sigma}{\underset{\alpha}{P}}_k\, g_{is}R^i_m\underset{\sigma}{v}^m = \overset{\sigma}{\underset{\alpha}{P}}_k\, g_{is}g^{ip}R_{mp}\underset{\sigma}{v}^m =$$

$$= \overset{\sigma}{\underset{\alpha}{P}}_k\, \delta^p_s R_{mp}\underset{\sigma}{v}^m = \overset{\sigma}{\underset{\alpha}{P}}_k\, R_{ms}\underset{\sigma}{v}^m = \nabla_k \underset{\alpha}{w}_s.$$

From (2), (11) and $\nabla_k \underset{\alpha}{v}_s = \overset{\sigma}{\underset{\alpha}{S}}_k\, \underset{\sigma}{w}_s$ we find:

$$\nabla_k \underset{\alpha}{w}_s = \overset{\sigma}{\underset{\alpha}{S}}_k\, g_{is}\underset{\sigma}{w}^i = \overset{\sigma}{\underset{\alpha}{S}}_k\, g_{is}R^i_p\underset{\sigma}{v}^p = \overset{\sigma}{\underset{\alpha}{S}}_k\, g_{is}g^{im}R_{mp}\underset{\sigma}{v}^p = \overset{\sigma}{\underset{\alpha}{S}}_k\, \delta^m_s R_{mp}\underset{\sigma}{v}^p = \overset{\sigma}{\underset{\alpha}{S}}_k\, R_{sm}\underset{\sigma}{v}^m$$

or

$$\overset{\sigma}{\underset{\alpha}{S}}_k = \overset{\sigma}{\underset{\alpha}{P}}_k\,.$$

Following [2] and [3] we define:

Definition 2. *We shall call a net* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in V_n$ *equidistant one if*

$$\nabla \left[k \sum_{\alpha=1}^{n} \underset{\alpha}{v}_i\right] = 0. \tag{13}$$

Proposition 2. *R-corresponding nets* $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v})$ *and* $(\underset{1}{w}, \underset{2}{w}, \ldots, \underset{n}{w})$ *are equidistant* *the in Riemannian space* $\mathbf{V}_n$ *if and only if*

$$\sum_{\alpha=1}^{n} \overset{\sigma}{\underset{\alpha}{T}}[{}_k \underset{\sigma}{v}{}_i] \sum_{\alpha=1}^{n} P[{}_k \overset{\sigma}{\underset{\alpha}{R}}{}_i] m \underset{\sigma}{v}{}^m = 0.$$

Proof. From (12) and (13) follows that the net $(\underset{1}{v}, \underset{2}{v}, \ldots, \underset{n}{v}) \in V_n$ is equidistant if and only if the first equation of (14) holds. The net $(\underset{1}{w}, \underset{2}{w}, \ldots, \underset{n}{w}) \in V_n$ is equidistant if and only if

$$\nabla \left[k \sum_{\alpha=1}^{n} \underset{\alpha}{w}{}_i \right] = 0.$$

Hence and from $\nabla_k w_s = \overset{\sigma}{\underset{\alpha}{P}}{}_k R_{ms} \underset{\sigma}{v}{}^m$ it follows that the net $(\underset{1}{w}, \underset{2}{w}, \ldots, \underset{n}{w}) \in V_n$ is equidistant if and only if the second equation of (14) holds.

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R-СЪОТВЕТНИ МРЕЖИ В *n*-МЕРНО РИМАНОВО ПРОСТРАНСТВО V_n

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Използвайки тензора на Ричи въвеждаме R-съответни мрежи в n-мерно риманово пространство. С помощта на намерените връзки между коефициентите от деривационните формули получаваме инвариантни характеристики на ортогонални и равнопътни мрежи.