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ON THE MEASURABILITY OF SETS OF SPHERES IN THE SIMPLY ISOTROPIC SPACE*

Adriyan Varbanov Borisov, Margarita Georgieva Spirova

The measurable sets of spheres and the corresponding invariant densities with respect to the group of the general similitudes and some its subgroups are described.

1. Introduction. The simply isotropic space ${I_3}^{(1)}$ is defined (see [5], [7], [8]) as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω and two complex conjugate straight lines f_1, f_2 into ω with a real intersection point F. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of the general simply isotropic similitudes. Passing on to affine coordinates (x, y, z) we have for G_8 the representation [5; p. 3]

$$
(1) \qquad \begin{aligned}
\overline{x} &= a + p(x\cos\varphi - y\sin\varphi),\\
\overline{y} &= b + p(x\sin\varphi + y\cos\varphi),\\
\overline{z} &= c + c_1x + c_2y + c_3z,
\end{aligned}
$$

where $p > 0, \varphi, a, b, c, c_1, c_2$ and $c_3 \neq 0$ are real parameters.

The d-distance between two nonparallel points and the s-distance between two parallel points in ${I_3}^{(1)}$ are relative invariants of the group G_8. We shall consider with G_8 and some its subgroups:

(i) $p = 1$ – the subgroup $B_7 \subset G_8$ of the simply isotropic similitudes of the d-distance [5; p. 5].

(ii) $c_3 = 1$ – the subgroup $S_7 \subset G_8$ of the simply isotropic similitudes of the s-distance [5; p. 6].

(iii) $c_3 = p$ – the subgroup $W_7 \subset G_8$ of the simply isotropic angulur similitudes [5; p. 16].

(iv) $\varphi = 0$ – the subgroup $G_7 \subset G_8$ of the boundary simply isotropic similitudes [5; p. 8].

(v) $G_6 = G_7 \cap W_7$ – the subgroup of the volume preserving boundary simply isotropic similitudes [5; p. 8].

(vi) $B_6 = B_7 \cap G_7$ – the subgroup of the modular boundary motions [5; p. 9].

(vii) ${B_6}^{(1)} = B_7 \cap S_7$ – the subgroup of the simply isotropic motions [5; p. 7].

(viii) $p = 1, \varphi = 0, c_3 = 1$ – the subgroup B_5 of the unimodular boundary motions [5; p. 9].

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We study the measurability in the sense of M. I. Stoka [6], G. I. Drinfel'd and A. V. Lucenko [2]–[4] of sets of spheres with respect to G_8 and indicated above subgroups.

2. Measurability with respect to G_8. Let be given in the space ${I_3}^{(1)}$ a quadric $\sum$ whose equation has the form

$$x^2 + y^2 + 2\alpha x + 2\beta y + 2\gamma z + \delta = 0, \tag{2}$$

where α, β, γ and δ are real parameters. We note [5; p.67] that depending on $\gamma \neq 0$ or $\gamma = 0$ the quadric $\sum$ is a sphere of parabolic type or a sphere of cylindrical type, respectively. Under the action of (1) the quadric $\sum(\alpha, \beta, \gamma, \delta)$ is transformed into the quadric $\overline{\sum}(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta})$ as

$$\begin{aligned} \overline{\alpha} &= -a + p{c_3}^{-1}[(\alpha c_3 - \gamma c_1)\cos\varphi - (\beta c_3 - \gamma c_2)\sin\varphi], \\ \overline{\beta} &= -b + p{c_3}^{-1}[(\alpha c_3 - \gamma c_1)\sin\varphi + (\beta c_3 - \gamma c_2)\cos\varphi], \\ \overline{\gamma} &= p^2{c_3}^{-1}\gamma, \\ \overline{\delta} &= a^2 + b^2 + \delta p^2 - 2p{c_3}^{-1}\{\gamma c p + [(\alpha c_3 - \gamma c_1)a + (\beta c_3 - \gamma c_2)b]\cos\varphi - \\ &\quad -[(\beta c_3 - \gamma c_2)a - (\alpha c_3 - \gamma c_1)b]\sin\varphi\}. \end{aligned} \tag{3}$$

The transformations (3) form the so-called associated group $\overline{G_8}$ of G_8 [6 ;p.34]. $\overline{G_8}$ is isomorphic to G_8 and the invariant density with respect to G_8 of the quadrics (2), if it exists, coicides with the invariant density with respect to $\overline{G_8}$ of the points $(\alpha, \beta, \gamma, \delta)$ in the set of parameters [6; p.33]. The infinitesimal operators of $\overline{G_8}$ are

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial\alpha} + 2\alpha\frac{\partial}{\partial\delta}, \ Y_2 = \frac{\partial}{\partial\beta} + 2\beta\frac{\partial}{\partial\delta}, \ Y_3 = \gamma\frac{\partial}{\partial\delta}, \\ &Y_4 = \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + 2\gamma\frac{\partial}{\partial\gamma} + 2\delta\frac{\partial}{\partial\delta}, \ Y_5 = \beta\frac{\partial}{\partial\alpha} - \alpha\frac{\partial}{\partial\beta}, \\ &Y_6 = \gamma\frac{\partial}{\partial\alpha}, \ Y_7 = \gamma\frac{\partial}{\partial\beta}, \ Y_8 = \gamma\frac{\partial}{\partial\gamma}. \end{aligned} \tag{4}$$

We destinguish the following cases:

Case I. $\gamma \neq 0$, i.e. $\sum$ is a sphere of parabolic type. We can write

$$Y_4 = 2\frac{\delta}{\gamma}Y_3 + \frac{\alpha}{\gamma}Y_6 + \frac{\beta}{\gamma}Y_7 + 2Y_8.$$

Since the infinitesimal operators Y_3, Y_6, Y_7 and Y_8 are arcwise unconnected and

$$Y_3\left(2\frac{\delta}{\gamma}\right) + Y_6\left(\frac{\alpha}{\gamma}\right) + Y_7\left(\frac{\beta}{\gamma}\right) + Y_8(2) \neq 0,$$

then it follows [2]–[4] that set (2) of sphere of parabolic type is not measurable under G_8 and it has not measurable subsets.

Case II. $\gamma = 0$, i.e. $\sum$ is a sphere of cylindrical type. Now the infinitesimal operators has the form

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial\alpha} + 2\alpha\frac{\partial}{\partial\delta}, \ Y_2 = \frac{\partial}{\partial\beta} + 2\beta\frac{\partial}{\partial\delta}, \ Y_3 = 0, \ Y_4 = \alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + 2\delta\frac{\partial}{\partial\delta}, \\ &Y_5 = \beta\frac{\partial}{\partial\alpha} - \alpha\frac{\partial}{\partial\beta}, \ Y_6 = 0, \ Y_7 = 0, \ Y_8 = 0. \end{aligned}$$

Obviously Y_1, Y_2 and Y_4 are arcwise unconnected and $Y_5 = \beta Y_1 - \alpha Y_2$. But $Y_1(\beta) -$

$Y_2(\alpha) = 0$ and the corresponding associated group $\overline{G_8}$ is measurable and the integral invariant function $f = f(\alpha, \beta, \delta)$ satisfies the system of R. Deltheil [1; p.28], [6; p.11] $Y_1(f) = 0,\ Y_2(f) = 0,\ Y_4(f) + 4f = 0$. The system has the solution

$$f = c(\alpha^2 + \beta^2 - \delta)^{-2},$$

where $c = const$.

We summarize the foregoing results in

Theorem 1. *(i) The set of spheres of parabolic type is not measurable under G_8 and it has not measurable subsets.*

(ii) The set of spheres of cylindrical type

$$\Sigma : x^2 + y^2 + 2\alpha x + 2\beta y + \delta = 0 \tag{5}$$

is measurable with respect to the group G_8 and has the invariant density

$$d\Sigma = (\alpha^2 + \beta^2 - \delta)^{-2} d\alpha \wedge d\beta \wedge d\delta. \tag{6}$$

***Remark* 1.** If we denote by $r = \sqrt{\alpha^2 + \beta^2 - \delta}$ the radius of the sphere of cylindrical type (5) and by $Q(x_0 = -\alpha, y_0 = -\beta, 0)$ the center of the Euclidean circle

$$k:\ x^2 + y^2 + 2\alpha x + 2\beta y + \delta = 0, z = 0$$

into the coordinate plane Oxy, then the density (6) can be written in the form

$$d\Sigma = 2r^{-3} dQ \wedge dr,$$

where $dQ = dx_0 \wedge dy_0$ is the Euclidean density of the points in the plane Oxy.

3. Measurability with respect to the subgroups of G_8. The associated group $\overline{B_7}$ of the group B_7 has the infinitesimal operators $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ and Y_8 in (4).

Case I. $\gamma \neq 0$. Now

$$Y_1 = 2\frac{\alpha}{\gamma} Y_3 + \frac{1}{\gamma} Y_6,\ Y_2 = 2\frac{\beta}{\gamma} Y_3 + \frac{1}{\gamma} Y_7,\ Y_5 = 2\frac{\beta}{\gamma} Y_6 - \frac{\alpha}{\gamma} Y_7,$$

where Y_3, Y_6, Y_7 and Y_8 are arcwise unconnected. It is easy to verify that the integral invariant function $f = f(\alpha, \beta, \gamma, \delta)$ satisfies the system of R. Deltheil

$$Y_3(f) = 0,\ Y_6(f) = 0,\ Y_7(f) = 0, Y_8(f) + f = 0$$

and therefore $f = c\gamma^{-1}$, where $c = const$.

Case II. $\gamma = 0$. Now

$$Y_3 = 0,\ Y_6 = 0,\ Y_7 = 0,\ Y_8 = 0,\ Y_5 = \beta Y_1 - \alpha Y_2$$

and consequently the group $\overline{B_7}$ acts intransitively on the set (5), i.e. the set (5) is not measurable with respect to the group B_7. From $Y_1(\beta) + Y_2(-\alpha) = 0$ and $Y_1(f) = 0$, $Y_2(f) = 0$ we deduce that the set (6) has the measurable subset

$$\alpha^2 + \beta^2 - \delta = h, h = const.$$

Thus we have the following

Theorem 2. *(i) The set of spheres of parabolic type (2) is measurable with respect to the group B_7 and has the invariant density*

$$d\Sigma = |\gamma|^{-1} d\alpha \wedge d\beta \wedge d\gamma \wedge d\delta. \tag{7}$$

(ii) The set of the spheres of cylindrical type (5) is not measurable with respect to the group B_7. It has the measurable subset

$$\alpha^2+\beta^2-\delta=h, h=const$$

with the invariant density $d\Sigma = d\alpha\wedge d\beta$.

Remark 2. Let us denote by $R=-\dfrac{1}{2\gamma}$ and by

$$Q\left(x_0=-\alpha, y_0=-\beta, z_0=\frac{\alpha^2+\beta^2-\delta}{2\gamma}\right)$$

the radius and the vertex of the sphere of parabolic type (2), respectively. Then the formula (7) becomes

$$d\Sigma = R^{-2}dR\wedge dQ,$$

where $dQ = dx_0\wedge dy_0\wedge dz_0$ is the invariant density of the points in ${I_3}^{(1)}$ under the group ${B_6}^{(1)}$.

By arguments similar to the ones used above we examine the measurability of the set of spheres (2) with respect to all the rest groups. We collect the results in the following table:

group	parabolic type	cylindrical type
S_7	$d\Sigma = \gamma^{-6}d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta$	$d\Sigma = (\alpha^2+\beta^2-\delta)^{-2}d\alpha\wedge d\beta\wedge d\delta$
W_7	is not measurable and has not measurable subsets	$d\Sigma = (\alpha^2+\beta^2-\delta)^{-2}d\alpha\wedge d\beta\wedge d\delta$
G_7	is not measurable and has not measurable subsets	$d\Sigma = (\alpha^2+\beta^2-\delta)^{-2}d\alpha\wedge d\beta\wedge d\delta$
G_6	$d\Sigma = \lvert\gamma\rvert^{-\frac{7}{3}}d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta$	$d\Sigma = (\alpha^2+\beta^2-\delta)^{-2}d\alpha\wedge d\beta\wedge d\delta$
B_6	$d\Sigma = \lvert\gamma\rvert^{-1}d\alpha\wedge d\beta\wedge d\gamma\wedge d\delta$	it is not measurable, but has the measurable subset $\alpha^2+\beta^2-\delta=h$ ($h=const$) with $d\Sigma = d\alpha\wedge d\beta$
${B_6}^{(1)}$	it is not measurable, but has the measurable subset $\gamma=h$ ($h=const$) with $d\Sigma = d\alpha\wedge d\beta\wedge d\delta$	it is not measurable, but has the measurable subset $\alpha^2+\beta^2-\delta=h$ ($h=const$) with $d\Sigma = d\alpha\wedge d\beta$
B_5	it is not measurable, but has the measurable subset $\gamma=h$ ($h=const$) with $d\Sigma = d\alpha\wedge d\beta\wedge d\gamma$	it is not measurable, but has the measurable subset $\alpha^2+\beta^2-\delta=h$ ($h=const$) with $d\Sigma = d\alpha\wedge d\beta$

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Adriyan Varbanov Borisov
Dept. of Descriptive Geometry
Univ. of Architecture
Civil Eng. and Geodesy
1, Christo Smirnenski Blvd.
1421 Sofia, Bulgaria
e-mail: adribor_fgs@uacg.acad.bg

Margarita Georgieva Spirova
Fac. of Math. and Informatics
Shumen University
"Episkop Konstantin Preslavski"
115, Alen Mak Str.
9712 Shumen, Bulgaria
e-mail:margspr@fmi.shu-bg.net

ВЪРХУ ИЗМЕРИМОСТТА НА МНОЖЕСТВА ОТ СФЕРИ В ПРОСТО ИЗОТРОПНО ПРОСТРАНСТВО

Адриян В. Борисов, Маргарита Г. Спирова

Описани са измеримите множества от сфери и са намерени съответните им инвариантни гъстоти относно групата на общите подобности и някои нейни подгрупи.