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GAUSS INTEGERS AND DIOPHANTINE FIGURES

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The study of Diophantine figures in the plane (see [1–4]) involves different geometric and number-theoretic notions. This paper gives a survey without proofs on the obtained up to now results. Some new results and problems are included.

Gauss integers.

General properties, unities, associate element. It is well known that the ring $\mathbb{Z}[i]$ is an Euclidean domain with respect to the norm $\mathbf{N}(n+im) = n^2+m^2$. The norm is a function of the type $\mathbb{Z}[i] \to \mathbf{N}$, such that for every two elements a and b in $\mathbb{Z}[i]$, with b different from zero, there are two elements c and d in $\mathbb{Z}[i]$ for which $a = cb + d$ and in the case $d \neq 0$ we have $\mathbf{N}(d) < \mathbf{N}(b)$.

An element $a + ib$ is an unit in $\mathbb{Z}[i]$ if there is $x + iy \in \mathbb{Z}[i]$ such that $(a + ib)(x + iy) = 1$, $1 = 1 + i.0$. It is not difficult to see that there are only four units in $\mathbb{Z}[i]$, namely $1, -1, i, -i$. We say that $\beta \in \mathbb{Z}[i]$ is an associate element of $\alpha \in \mathbb{Z}[i]$ if $\beta = \varepsilon\alpha$, where ε is an unit in $\mathbb{Z}[i]$. Clearly if β is an associate to α, then α is an associate to β. Every $\alpha \in \mathbb{Z}[i]$ has four associate elements $1.\alpha, (-1).\alpha, i.\alpha, (-i).\alpha$.

Parity in $\mathbb{Z}[i]$: Even and odd gauss integers.

Definition. *We say that the Gauss integers $n + im$, $n, m \in \mathbb{Z}$, is an even integer in $\mathbb{Z}[i]$ if n and m are of the same parity in $\mathbb{Z}$. If n and m are of different parity in $\mathbb{Z}$, we say that $n + im$ is an odd Gauss integer in $\mathbb{Z}[i]$.*

Proposition 1. *If the Gauss integer $n + im$ is even in $\mathbb{Z}[i]$ there is $p + iq \in \mathbb{Z}[i]$ such that $n + im = 2(p + iq)$ or $n + im = 2(p + iq) + 1 + i$. Respectively, if $n + im$ is odd in $\mathbb{Z}[i]$ there is $r + is$ such that $n + im = 2(r + is) + 1$ or $n + im = 2(r + is) + i$*

Proposition 2. *The Gauss integer $n+im$ is even in $\mathbb{Z}[i]$ iff $n+im \equiv 0 \,(\mathrm{mod}\ (1+i))$, respectively $n + im$ is odd in $\mathbb{Z}[i]$ iff $n + im \equiv 1 \,(mod\ (1 + i))$.*

Proposition 3. *The Gauss integer $n + im$ is even in $\mathbb{Z}[i]$ iff the norm $\mathbf{N}(n + im) = n^2 + m^2$ is even in $\mathbb{Z}$. Respectively, $n + im$ is odd in $\mathbb{Z}[i]$ iff the norm $\mathbf{N}(n + im)$ is odd in $\mathbb{Z}$.*

It is not difficult to prove that the above three propositions are equivalent.

Arithmetic properties of gauss integers. It is easy to prove (with the help of the above stated propositions) that: **the sum of Gauss integers** of common parity is even, and the sum of Gauss integers of different parity is odd, **the product of Gauss integers** satisfy the same rules as in $\mathbb{Z}$, $(even) \times (even) = (even)$, $(odd) \times (odd) = (odd)$,

$(even) \times (odd) = (even)$. **The square** of an even Gauss integer is an even Gauss integer, respectively the square of an odd Gauss integer is an odd Gauss integer. More precisely, if α is even in $\mathbb{Z}[i]$, then $\alpha = (1+i)^2\alpha_2 = 4\alpha_1$. If β is odd in $\mathbb{Z}[i]$, then $\beta^2 = (1+i)^2\beta_2+1 = 4\beta_1 + 1$.

Square radical of a Gauss integer (S. Dimiev): Let $\alpha = m + im$ be a Gauss integer with $m \neq 0$. We consider the equation $z^2 = \alpha$ with $z \in \mathbf{C}$. Each solution of this equation is called square radical of α. In the case that there is $l \in \mathbf{N}$ such that (n, m, l) to be a Pythagorean triple in $\mathbb{Z}$, i.e. $n^2 + m^2 = l^2$, we have

$$\sqrt{\frac{n+im}{2}} = \frac{t + i\frac{m}{t}}{2}, \quad t^2 = n + l$$

Proof and some discussions are given by P. Guncheva [10].

Indecomposable gauss integers.

Definition ([6]). *An element α of $\mathbb{Z}[i]$ is called **indecomposable** Gauss integer or **prime** Gauss integer if it is impossible to present it as product of two elements $\lambda, \mu \in \mathbb{Z}[i]$, both of which are not units, i.e. different from $1, -1, i, -i$.*

Below we shall give examples of Gauss prime integers. The norm of a Gauss integer is sum of two squares, i.e. if $\alpha = n + im$, $n, m \in \mathbb{Z}$, then $\mathbf{N}(\alpha) = n^2 + m^2$. With this in mind we set $\mathbf{N}(\alpha) \equiv t \,(\text{mod } 4)$, where $t = 0, 1, 2, 3$. It is not difficult to see that the case $t = 3$ is impossible. More precisely, we have: $\mathbf{N}(\alpha) = n^2 + m^2 \neq 4s + 3, \quad s \in \mathbb{Z}$. So, in the case $\mathbf{N}(\alpha)$ is odd, it follows $\mathbf{N}(\alpha) = 4s + 1$, or $\mathbf{N}(\alpha) \equiv 1 \,(\text{mod } 4)$. In the next exposition we need two well known theorems from the number theory in $\mathbb{Z}$.

The first one is a theorem of Fermat (see Edwards [9]): *each prime number in $\mathbb{Z}$ such that $p \equiv 1 \,(mod\ 4)$ can be written as a sum of two squares, i.e. there exist $a, b \in \mathbb{Z}$ such that $p = a^2 + b^2$. When a, b are positive integers a is odd, b is even, the above mentioned representation is unique.* The second theorem asserts: *if $(x, y) = 1$ then $x^2 + y^2$ has at least one prime divisor of the form $4k + 1$* see for instance [7], pp. 56–102).

Now we turn to the description of the prime Gauss integers. The mentioned description is based on the comparison with prime integers in $\mathbb{Z}$. We will see that some prime integers in $\mathbb{Z}$ are decomposable in $\mathbb{Z}[i]$ and that all prime Gauss integers are divisors of prime integers in $\mathbb{Z}$. More precisely we have the following (suggested by [6])

Lemma 1. *If α is a prime element of $\mathbb{Z}[i]$, then there is a prime $p \in \mathbb{Z}$ such that $\alpha | p$.*

Lemma 2. *If α is a Gauss integer and $\mathbf{N}(\alpha)$ is a prime integer in $\mathbb{Z}$, then α is prime Gauss integer.*

Based on the proved two lemmas we give examples of prime Gauss integer and decomposable in $\mathbb{Z}[i]$ prime integers in $\mathbb{Z}$. These are $1+i$ in $\mathbb{Z}[i]$ and 2 in $\mathbb{Z}$. Indeed $\mathbf{N}(1+i) = 2$ and we apply Lemma2; $2 = (-i)(1 + i)^2$. Lemma 1 gives $(1 + i)|2$.

Theorem (*suggested by* [6]). *The indecomposable elements in $\mathbb{Z}[i]$ are the following:*

a) *All prime integers in $\mathbb{Z}$ of the form $4k + 3$ and all their associate elements in $\mathbb{Z}[i]$;*

b) *The number $1 + i$ and its associates;*

c) *If p is a prime integer in $\mathbb{Z}$ of the form $4k + 1$ and $\mathbf{N}(\alpha) = p$, $\alpha = x + iy$, $x, y \in \mathbb{Z}$, i.e. $p = x^2 + y^2$, then α and $\overline{\alpha}$ are indecomposable elements in $\mathbb{Z}[i]$ with all their associates.*

d) *There are no other indecomposable elements in* $\mathbb{Z}[i]$*, more precisely if* $\mathbf{N}(\alpha) = q$*, where* q *is not prime integer in* $\mathbb{Z}$*, then* α *is indecomposable element in* $\mathbb{Z}[i]$*.*

Gauss-pythagorean integers.

Definition (S. Dimiev)**.** *The Gauss integer* $\alpha = x+iy$*,* $x, y \in \mathbb{Z}$*, is said to be a Gauss-Pythagorean number if there exists* $z \in \mathbb{Z}$ *such that the triple* x, y, z *be a Pythagorean triple:* $x^2 + y^2 = z^2$*.*

We shall denote the set of all Gauss-Pythagorean integers by $\mathbb{GP}[i]$, $\mathbb{GP}[i] \subset \mathbb{Z}[i]$. The zero element and the units of $\mathbb{Z}[i]$ are not Gauss-Pythagorean integers. We remark that the sum of two Gauss-Pythagorean numbers is not Gauss-Pythagorean number in general, but it is easy to prove that the product of two Gauss-Pythagorean numbers is always a Gauss-Pythagorean integer. So $\mathbb{GP}[i]$ is a multiplicative subsemigroup of the multiplicative group $\mathbb{Z}[i] - \{0\}$ of the ring $\mathbb{Z}[i]$.

The conjugate and the associate elements of a Gauss-Pythagorean element are Gauss-Pythagorean integers too. The next Lemma is useful in the following exposition

Lemma (K. Markov)**.** *Let* $\alpha = x + iy$ *be a Gauss-Pythagorean number. Then there exists a Gauss integer* τ *such that* $\mathbf{N}(\alpha) = (\mathbf{N}(\tau))^2$*.*

Corollary. *There is no element in* **GP** $[i]$ *which is prime Gauss integer.*

Definition. *An element* $\tau \in \mathbb{GP}[i]$ *is said to be a prime Gauss-Pythagorean integer if it is impossible to represent it as a product of two elements of* $\mathbb{GP}[i]$*.*

Theorem. *There exists an infinite number of indecomposable Gauss-Pythagorean numbers.*

A proof of this theorem is due to K. Markov.

Primitive triples in $\mathbb{Z}[i]$**.**

Definition. *We say that the triple* α, β, γ *of Gauss integers is a* ***primitive triple*** *if the unique common divisors of the elements of the triple are the unities in* $\mathbb{Z}[i]$*.*

We denote this by $(\alpha, \beta, \gamma) = 1$. Like in $\mathbb{Z}$ we shall write $(\alpha, \beta) = 1$, when α and β satisfy the same condition. Ordinary by (α, β, γ) it is denoted the GCD of α, β and γ. Analogously
(α, β) is the GCD of α and β. It is easy to see that
$(\alpha, \beta, \gamma)=((\alpha, \beta), \gamma)=(\alpha, (\beta, \gamma))=((\alpha, \gamma), \beta)$. If $\overline{\alpha}$ is the complex conjugate of α we have:
$(\alpha, \beta)=\delta$ iff $(\overline{\alpha}, \overline{\beta})=\overline{\delta}$ in particular $(\alpha, \beta)=1$ iff $(\overline{\alpha}, \overline{\beta})=1$. Analogously $(\alpha, \beta, \gamma)=\delta$ iff $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \overline{\delta}$ and in particular $(\alpha, \beta, \gamma) = 1$ iff $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = 1$. Clearly,
$(\alpha, \beta) = (\alpha, \gamma) = (\beta, \gamma) = 1$ implies $(\alpha, \beta, \gamma) = 1$, but the inverse is not true.

Proposition 4. $(\mathbf{N}(\alpha), \mathbf{N}(\beta), \mathbf{N}(\gamma)) = 1$ *in* $\mathbb{Z}$ *implies* $(\alpha, \beta, \gamma) = 1$ *in* $\mathbb{Z}[i]$*.*

The proof is not difficult. *Example:* $(3 + i, 2 + i, 8 + i) = 1$, but $(\mathbf{N}(3 + i), \mathbf{N}(2 + i), \mathbf{N}(8 + i)) = (10, 5, 65) = 5$.

We remark that $(3 + i, 2 + i) = 1$ does not imply $(\mathbf{N}(3 + i), \mathbf{N}(2 + i)) = 1$.

Pythagorean triples, primitive pythagorean triples. In a primitive Pythagorean triple in $\mathbb{Z}[i]$ at least one element is odd. Let α, β, γ be a primitive Pythagorean triple, i.e. $\alpha^2 + \beta^2 = \gamma^2$ and $(\alpha, \beta, \gamma) = 1$. There are two possibilities for γ : the first one is γ to be even. In this case we have: $(odd)^2 + (odd)^2 = (even)^2$. The second possibility is γ to be odd. So we have for instance: $(even)^2 + (odd)^2 = (odd)^2$.

Proposition 4 obtains for Pythagorean triples the following stronger form: Proposition **4′**. $(\mathbf{N}(\alpha), \mathbf{N}(\beta), \mathbf{N}(\gamma)) = 1$ in $\mathbb{Z}$ is equivalent to $(\alpha, \beta, \gamma) = 1$ in $\mathbb{Z}[i]$.

Proposition 5. *Formulae for primitive Pythagorean triples:*
$\alpha = 2\lambda\mu$, $\beta = \lambda^2 - \mu^2, \gamma = \lambda^2 + \mu^2, (\lambda, \mu) = 1, \lambda, \mu \in \mathbb{Z}\ [i]$
where λ *and* μ *are of different parity.*

For the proof we can imitate the known proof in Z having in mind the sense of the parity in Z[i] (roughly speaking replacing 2 by 1 + i).

Proposition 6 (Fermat Last Theorem in $\mathbb{Z}[i]$ for $n = 4$)**.** *The equation* $x^4 + y^4 = z^4$ *has no Gauss integer solutions.*

For the proof we propose a method of Fermat type descent with respect to the norm. This will be exposed elsewhere [11].

Diophantine figures. We shall consider the Cartesian plane $\mathbf{R} \times \mathbf{R}$ ($\mathbf{R}-$ the field of real numbers). A **complete Cartesian graph** is by definition the couple (**V,S**) where **V** is the set of points in $\mathbf{R} \times \mathbf{R}$, called vertices, and **S** is the set of all segments $[P.Q]$ with $P, Q \in \mathbf{V}$, $P \neq Q$. A **Cartesian Erdös graph** is by definition a Cartesian graph $(\mathbf{V}, \mathbf{S})$ for which the length of each segment in **S** is a integer number $\neq 0$. If the set of vertices **V** is infinite we shall say that $(\mathbf{V}, \mathbf{S})$ is an infinite graph.

Theorem (Erdös)**.** *The vertices of an infinite Cartesian Erdös graph are situated on a straight line in the Cartesian plane.*

Definition. *The Cartesian product* $\mathbf{Z} \times \mathbf{Z}$ *will be called* ***Diophantine plane****.*

Clearly $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$, or the Diophantine plane is the lattice of the points in $\mathbf{R} \times \mathbf{R}$ with integer coordinates. We will consider complete graphs in the Diophantine plane, i.e. the set of couples $(\mathbf{V}, \mathbf{S})$, $\mathbf{V} \subset \mathbb{Z} \times \mathbb{Z}$ and **S** is the same as above. A **Diophantine figure** is by definition a complete graph in the Diophantine plane for which the length of each of its segments is an integer number. Diophantine figures which contain at least three different non-collinear vertices will be considered. **Erdös-Diophantine figure** is by definition a **maximal** Diophantine figure, i.e. a Diophantine figure for which there is not a larger one.

The existence of Erdös-Diophantine figures follows from the above cited Erdös theorem. Indeed, according this theorem each increasing sequence of finite non-linear Diophantine figures $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$ stabilizes at some index $k_0 \in \mathbf{N}$. Then F_{k_0} is an Erdös-Diophantine figure.

Diophantine planimetry – examples. A closed path in a Diophantine figure F is defined by a sequence of vertices $P_0, P_1, \ldots, P_n = P_0$, $P_j \in F$, and the union of the connecting segments. For a Diophantine triangle there is only one closed path constituted from all vertices and all segments of the triangle.

Proposition 7. *The sum of lengths of the segments of a closed path in a Diophantine figure is an even integer.*

The proof can be derived by induction from the following

Lemma (M. Brancheva)**.** *The sum of lengths of the sides of a Diophantine triangle is always an even integer.*

This Lemma is a generalization of the analogous property of Pythagorean triangles.

Proposition 8 (M. Brancheva). *Let $\triangle ABX$ lie in the Diophantine plane and (a_1, a_2), (b_1, b_2), (x_1, x_2) are the coordinates of the vertices A, B, X resp. Let suppose $a_1 = a_2 = 0$; let the lengths of the segments of the triangle be: $|AB| = \mathbf{c}$, $|BC| = \mathbf{a}$, $|AC| = \mathbf{b}$. For given lengths $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and given coordinates (b_1, b_2) we have the following Diophantine equation of first degree for $x_1, x_2 : 2b_1x_1 + 2b_2x_2 = \mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2$.*

Remark. The above written Diophantine equation is not always solvable. Indeed, if a solution exists, the number $\mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2$ must be even, but according to Proposition 7 the same is true for the number $\mathbf{b}^2 + \mathbf{c}^2 + \mathbf{a}^2$. This implies the following necessary condition: the number $\mathbf{b}^2 + \mathbf{c}^2$ must be even. We see that the classical construction is not always possible for Diophantine triangles.

Diophantine triangles: classification. Each Diophantine triangle can be inscribed in a uniquely determined rectangle with sides parallel to the coordinate axes. This enveloping rectangle help us to find the next

Lemma (Classification lemma). *There are 4 essentially different types of Diophantine triangles (see the Figures below): (1) Pythagorean triangle; (2) and (3) obtained from two Pythagorean triangles with common cathetus; (4) new kind of Diophantine triangle.*

The proof is obtained by simple examination of the possibilities for the disposition of the vertices of the inscribed triangle on the sides of the enveloping rectangle.

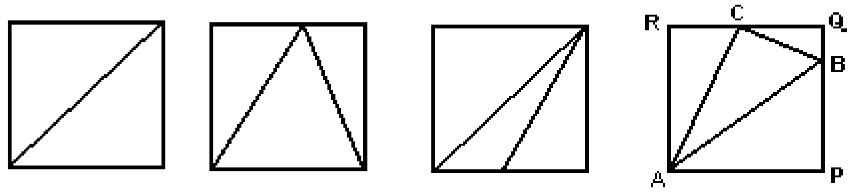

Figure 1

Remark. The supplementary part of the enveloping rectangle with respect to the inscribed Diophantine triangle is composed by 1, 2 or 3 Pythagorean triangles.

The Classification lemma suggests different calculations. We consider the case (4).

Applying the well known formulae for Pythagorean triples we can write:

for $\triangle APB: \; AP = u^2 - v^2, BP = 2uv, AB = u^2 - v^2, \quad u > v$

for $\triangle BQC: \; CQ = p^2 - q^2, BQ = 2pq, BC = p^2 + q^2, \; p > q$

for $\triangle ACR: \; RC = x^2 - y^2, AR = 2xy, AC = x^2 + y^2, \quad x > y$

From $AR = PQ$ it follows: $2xy = 2pq + 2uv$ and from $AP = RQ$ it follows:

$u^2 - v^2 = x^2 - y^2 + p^2 - q^2$.

In the Gauss-Diophantine plane we have: $(x + iy)^2 = (q + ip)^2 + (u + iv)^2$.

This means that $(q + ip, u + iv, x + iy)$ is a Pythagorean triple in $\mathbb{Z}[i]$. Now we apply the Proposition 5, according to which: $x + iy = (a + ib)^2 + (c + id)^2, \quad a, b, c, d \in \mathbb{Z}$, $u + iv = (a + ib)^2 - (c + id)^2, \quad q + ip = 2(a + ib)(c + id)$.

Consequently:

$x = a^2 - b^2 + c^2 - d^2$, $y = 2(ab - cd)$, $u = a^2 - b^2 - c^2 + d^2$, $v = 2(ab - cd)$, $p = 2(ad + bc)$, $q = 2(ac - bd)$.

Taking $b = c = d = 1$ we get only one parameter a. After calculations we obtain for $a = 4$ the following triple: $AB = 261, BC = 136, AC = 325$. For $a = 5$ we obtain another triple:$AB = 640, BC = 208, AC = 270$. It can be verified that these two triples define Diophantine triangles, and the supplementary triangles APB, BQC, CRA are Pythagorean.

$(AP = 189, PB = 180, BQ = 120, QC = 64, CR = 125, RA = 300)$.

Remark. The above exposed examples (proposed by N. Milev and M. Brancheva) give an idea how to proceed practically to get Diophantine triangles of the kind 4.

Diophantine figures composed by pythagorean triangles with common cathetus. The simplest Diophantine figures are composed by many Pythagorean triangles.

We shall consider the set of Pythagorean triples (x, n, z) in $\mathbf{N}$, i. e. $x^2 + n^2 = z^2$. It is not supposed that these triples are primitive. We introduce the function $\kappa : \mathbf{N} \to \mathbf{N}$, $n \longmapsto \kappa(n)$, where $\kappa(n)$ is the number of all Pythagorean triangles with cathetus n. By definition $\kappa(0) = 0$ and $\kappa(m) = 0$ if m is not a cathetus in a Pythagorean triangle.

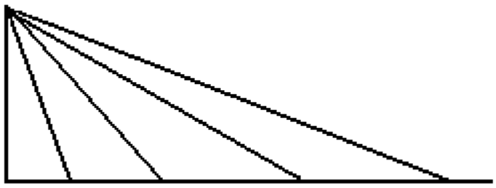

Figure 2

We shall write $d \mid n$ if d is a divisor of n. By $\eta(d)$ is denoted the set of all primitive Pythagorean triples with d as cathetus. It is clear that from each primitive Pythagorean triple (u, d, v) we can obtain a Pythagorean triple (x, n, y) with cathetus n. It is sufficient to multiply by $\frac{n}{d} = k$, i.e. $x = \frac{n}{d}u$, $y = \frac{n}{d}v$. Having in mind all divisors of n, we can introduce the following formula:

$$\kappa(n) = \sum_{d \mid n} \eta(d)$$

Lemma. *If $\delta(d)$ is the set of all divisors of d then $\eta(d) < \delta(d)$.*

Proof. with the help of the proposition 5.
Now let n be as follows: $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ and $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$ with $0 \le \beta_j \le \alpha_j$.
After some calculations we receive:

$$\kappa(n) < \delta(n) < \left(\prod_{j=1}^{n} (1 + \alpha_j) \right)^2$$

Theorem (M. Brancheva)**.** $\kappa(n) = O(n^{\varepsilon})$.

Proof. According to a well known formula (see [8]).

$$\delta(n) < \exp \left\{ (1 + \rho) \frac{\ln 2 \ln n}{\ln \ln(n)} \right\}$$

But $\exp\left\{(1+\rho)\dfrac{\ln 2 \ln n}{\ln\ln(n)}\right\} = n^{2(1+\rho)\ln 2/\ln\ln(n)}$. We receive:

$$[\delta(n)]^2 < n^{2(1+\rho)\ln 2/\ln\ln(n)}$$
$$[\delta(n)]^2 < o(n^{\varepsilon}), \varepsilon > 0.$$

Application. $\lim\limits_{n\to\infty}\dfrac{\kappa(n)}{n} = 0$ which is a conjecture of S. Dimiev.

It is proved in stronger form: $\lim\limits_{n\to\infty}\dfrac{\kappa(n)}{n^{\varepsilon}} = 0$ for every $\varepsilon > 0$.

Problems.

1. Let us denote by $\chi(l)$ the number of all Pythagorean triangles with hypotenuse $l, l \in \mathbf{N}$. Find the asymptotic of the function $\chi(l)$ when $l \to \infty$ following the above exposed case of the function $\kappa(n)$.

2. Given a Diophantine triangle ABC is it possible to find a point D in the Diophantine plane such that $ABCD$ to be a Diophantine figure? The case when there is no such point D means that the triangle ABC is an Erdös-Diophantine figure. Are there Erdös-Diophantine triangles? In the case when there is such a point D is it possible to find an effective algorithm of searching such points?

3. We say that the pyramid $ABCD$ is a Pythagorean-Diophantine pyramid if the coordinates of the vertices are integers, the lengths of each segment AB, AC, BC, BD, AD, CD are natural numbers and the triangles ADC, BDC and ADB are Pythagorean. Are there Pythagorean-Diophantine pyramids?

4. We say that the quaternion $q = n + im + jr + ks$, with $n, m, r, s \in \mathbb{Z}$, is a Hamiltonian integer. Each q is represented by a couple of Gauss integers as follows: $q = z + wj$, where $z = n + im$ and $w = r + is$. It is interesting to examine the possibility to develop a simiral theory for Hamiltonian integers.

5. Examine the coloring problem for Diophantine carpets. For a large class of such carpets the chromatic number is 2.

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REFERENCES

[1] M. N. Brancheva. Diophantine Figures and Diophantine Carpets, *Mathematics and Mathematical Education*, **30** (2001), 289-296.

[2] S. Dimiev. Physicalism in mathematics? *Mathematics and Mathematical Education*, **30** (2001), 78-85.

[3] S. Dimiev, I. Tonov. Diophantine Figures, *Mathematics and Mathematical Education*, **15** (1986), 383-390 (in Bulgarian).

[4] M. N. Brancheva, S. Dimiev, N. Milev. On Diophantine Figures, (to be published).

[5] M. N. Brancheva. Asymptotic for the function $\kappa(n)$ (submitted for publication)

[6] K. Ireland, M. Rosen. A Classical Introduction to Modern Number Theory, Springer Verlag, 1982.

[7] T. Tonkov. Figural Numbers (in bulgarian), Sofia, 1971.

[8] K. Prachar. Primzahlverteilung, Springer Verlag, 1957.

[9] H. M. Edwards. Fermat's Last Theorem, Springer Verlag, New-York, 1977.

[10] P. Guncheva. Square Radicals of Gauss Integers (to be published).

[11] S. Dimiev, K.Markov, M. Yawata. On the Geometry of Pythagorean Triples (to be published).

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ГАУСОВИ ЦЕЛИ ЧИСЛА И ДИОФАНТОВИ ФИГУРИ

Станчо Димиев, Красимир Марков

Тази статия има обзорен характер. Излагат се получените до сега резултати по аритметика на целите гаусови числа и приложенията им за конструирането на диофантови фигури.