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TRANSFORMATIONS OF THE SHAPE SPHERE*

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Two representations of the classes of similar triangles are the extended plane and the shape sphere. Using the shape theorems for triangles we construct some bijective maps of the shape sphere onto itself preserving the first collision point and obtain the explicit equations of these maps.

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1. Introduction. The equivalence classes of triangles with respect to the similarity transformations of the Euclidean plane to itself can be expressed by the points in the extended plane $\widetilde{\mathcal{E}}^2 = \mathbb{C} \cup \infty$. The main advantage of this representation is the first and second shape theorems for triangles proved by J. Lester in [3]. These theorems define two correspondences of the extended plane to itself: three-to-one and four-to-one. Fixing four or five points, we may consider one-to-one correspondences of $\widetilde{\mathcal{E}}^2$ onto itself. It is well-known that the complex projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ is homeomorphic to the two - dimensional sphere (see [1, ch. 4]). But the two-dimensional sphere $\mathrm{S}^2 \subset \mathbb{R}^3$ with radius $\frac{1}{2}$ (so called a shape sphere) gives another representation of the classes of similar triangles. Using a conformal map of the extended plane onto the shape sphere we can construct certain bijective transformation of the shape sphere. We observe that the considered transformations are conformal mappings preserving the first collision point and derive their explicit equations.

2. Classes of similar triangles. The extended plane $\mathbb{R}^2 \cup \infty$, or equivalently the complex projective line $\mathbb{C} \cup \infty$, can be used for a representation of the equivalence classes of triangles with respect to direct similarities in the plane. J. Lester applied this representation for the study of the Euclidean plane (see [3] and [4]). We recall some basic definitions and assertion from her complex analytic formalism. Identify the Eclidean plane $\mathbb{R}^2$ with the field of the complex numbers $\mathbb{C}$. Let $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{C}$ be three points and let at least two of them be different. Then, the number

$$\triangle_{\mathbf{pqr}} = \frac{\mathbf{p} - \mathbf{r}}{\mathbf{p} - \mathbf{q}} \in \mathbf{C} \cup \infty$$

is called a shape of the oriented triangle $\mathbf{pqr}$. It is clear that $\triangle_{\mathbf{pqr}} = \infty$ iff $\mathbf{p} = \mathbf{q}$. For any degenerate triangle with $\mathbf{p} \neq \mathbf{q}$, $\triangle_{\mathbf{pqr}} \in \mathbb{R}$. In particular: if $\mathbf{r}$ is a midpoint of the segment $\mathbf{pq}$, then $\triangle_{\mathbf{pqr}} = \dfrac{1}{2}$; if $\mathbf{q}$ is a midpoint of $\mathbf{pr}$, then $\triangle_{\mathbf{pqr}} = 2$ and if $\mathbf{p}$ is a

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midpoint of $\mathbf{qr}$, then $\triangle_{\mathbf{pqr}} = -1$. Similarly, $\mathbf{p} = \mathbf{r} \iff \triangle_{\mathbf{pqr}} = 0$ and $\mathbf{q} = \mathbf{r} \iff \triangle_{\mathbf{pqr}} = 1$. Thus, all degenerate isosceles triangles are described in terms of shapes. It is clear that the non-degenerate triangle $\mathbf{pqr}$ is right-handed iff $\operatorname{Im} \triangle_{\mathbf{pqr}} > 0$.

There is another way for a representation of classes of similar triangles (see [2]). The points of the two-sphere S^2 with radius $\frac{1}{2}$ also correspond to the equivalence classes of triangles with respect to the direct similarities in the plane. This sphere is called a shape sphere. The North and South pole, so called Lagrange's points L^+ and L^-, correspond to the classes of positive and negative oriented equilateral triangles. The points on the equator represent all classes of degenerate triangles.

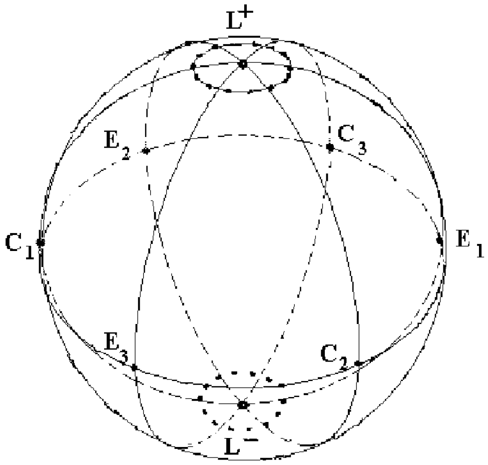

Fig. 1

Assume that the shape sphere $S^2 \subset \mathbb{R}^3$ is given by the equation

$$x_1^2 + x_2^2 + x_3^2 = \frac{1}{4}.$$

Then, we obtain the coordinates of all remarkable points on the shape sphere. For example, $L^+(0, 0, 1/2)$ and $L^-(0, 0, -1/2)$. The first Euler point $E_1(1/2, 0, 0)$ corresponds to the degenerate triangle $\mathbf{pqr}$ in which $\mathbf{r}$ is a midpoint of the segment $\mathbf{pq}$. The second and third Euler points $E_2\left(-\frac{1}{4}, \frac{\sqrt{3}}{4}, 0\right)$ and $E_3\left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}, 0\right)$ correspond to the casses in which $\mathbf{p}$ and $\mathbf{q}$ are midpoints of the segments determined by the remaining two points. The antipodal points to the Euler points are the collision points $C_1\left(-\frac{1}{2}, 0, 0\right)$, $C_2\left(\frac{1}{4}, -\frac{\sqrt{3}}{4}, 0\right)$ and $C_3\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, 0\right)$ corresponding to the degenerate triangles with coinsidences $\mathbf{p}$=$\mathbf{q}$, $\mathbf{q}$=$\mathbf{r}$ and $\mathbf{r}$=$\mathbf{p}$, respectively (see Figure 1). Obviously, the points on the semisphere

$$x_1^2 + x_2^2 + x_3^2 = \frac{1}{4}, \qquad x_3 > 0,$$

represent all non-degenerate and right-handed triangles.

Now, we consider a map σ of the extended plane onto the shape sphere with the following property: if $\widetilde{\mathcal{E}}^2 \ni \triangle \xrightarrow{\sigma} U \in \mathrm{S}^2$, then $\triangle$ and U represent the same class of similar triangles. Suppose that the coordinate plane $\{x_3 = 0\} \subset \mathbb{R}^3$ coincides with $\mathbb{R}^2 \cong \mathbf{C}$. Then the map $\sigma : \widetilde{\mathcal{E}}^2 \longrightarrow \mathrm{S}^2$ is represented by

$$\mathbf{C} \ni \triangle = (x, y) \xrightarrow{\sigma} U = (u_1, u_2, u_3) \in \mathrm{S}^2 \setminus C_1 \text{ and } \infty \xrightarrow{\sigma} C_1,$$

where

$$\begin{aligned} u_1 &= \frac{2x - 2x^2 - 2y^2 + 1}{4(x^2 + y^2 + 1 - x)} \\ u_2 &= \frac{\sqrt{3}(1 - 2x)}{4(x^2 + y^2 + 1 - x)} \\ u_3 &= \frac{2\sqrt{3}y}{4(x^2 + y^2 + 1 - x)}. \end{aligned} \tag{1}$$

The reverse map $\sigma^{-1} : \mathrm{S}^2 \longrightarrow \widetilde{\mathcal{E}}^2$ is defined by

$$\mathrm{S}^2 \setminus C_1 \ni U = (u_1, u_2, u_3) \xrightarrow{\sigma^{-1}} \triangle = (x, y) \in \mathbf{C} \text{ and } C_1 \xrightarrow{\sigma^{-1}} \infty,$$

where

$$x = \frac{1 + 2u_1 - 2\sqrt{3}u_2}{2(1 + 2u_1)}, \qquad y = \frac{\sqrt{3}u_3}{1 + 2u_1} \qquad (u_1 \neq -\frac{1}{2}). \tag{2}$$

We use the above formulas in the last section for obtaining of some mappings of the shape sphere onto itself.

3. Linear and linear-fractional transformations of the extended plane.

The main calculating tool in complex analytic formalism introduced by J. Lester is the first and second shape theorems for triangles (see [3]). We apply these theorems for obtaining of some mappings of the extended plane onto itself.

Suppose that $\mathbf{p}, \mathbf{q}, \mathbf{m_1}$ and $\mathbf{m_2}$ are four fixed points in the Euclidean plane such that $\mathbf{p} \neq \mathbf{q}, \mathbf{m_1} \neq \mathbf{m_2}$ and $\mathbf{p} \neq \mathbf{m_1}, \mathbf{m_2}$. Let $\mathbf{m_3}$ be an arbitrary point in the plane and let $\triangle_i$ $(i = 1, 2, 3)$ be the shape of the triangle $\mathbf{m}_i\mathbf{pq}$. Then $\triangle_i \neq 1$ for $i = 1, 2, 3$ and $\triangle_1 \neq \triangle_2$. Using the first shape theorem we calculate

$$\triangle_{\mathbf{m_1 m_2 m_3}} = \frac{(1 - \triangle_2)(\triangle_1 - \triangle_3)}{(1 - \triangle_3)(\triangle_1 - \triangle_2)} = \frac{(1 - \triangle_1)(1 - \triangle_2)}{\triangle_2 - \triangle_1} . \frac{1}{1 - \triangle_3} + \frac{1 - \triangle_2}{\triangle_1 - \triangle_2}.$$

But $\mathbf{m_1}$ and $\mathbf{m_2}$ are fixed and different points. Hence, the both complex numbers

$$a = \frac{(1 - \triangle_1)(1 - \triangle_2)}{\triangle_2 - \triangle_1} \text{ and } b = \frac{1 - \triangle_2}{\triangle_1 - \triangle_2}$$

are neither 0 or ∞. Setting

$$z = \triangle_{\mathbf{pqm_3}} = \frac{1}{1 - \triangle_{\mathbf{m_3 pq}}} = \frac{1}{1 - \triangle_3} \in \mathbf{C},$$

we obtain that $\triangle_{\mathbf{m_1 m_2 m_3}} = az + b$. Thus, we may consider the mapping φ of the extended plane onto itself defined by

$$\mathbf{C} \cup \infty \ni z = \triangle_{\mathbf{pqm_3}} \xrightarrow{\varphi} \triangle_{\mathbf{m_1 m_2 m_3}} = az + b \in \mathbf{C} \cup \infty.$$

It is clear that $\varphi(z) = \infty$ iff $z = \infty$, i. e. the restriction $\varphi|_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{C}$ is a similarity. If $z = x + y.i$, $\varphi(z) = z' = x' + y'.i$, $a = a_1 + a_2.i$ and $b = b_1 + b_2.i$, the restricted mapping

$\varphi|_{\mathbf{C}}$ is represented by the equations

$$
\begin{array}{rcl} x' & = & a_1x - a_2y + b_1 \\ y' & = & a_2x + a_1y + b_2. \end{array} \tag{3}
$$

Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{m_1}$ and $\mathbf{m_2}$ be five different fixed points in the Euclidean plane and let $\mathbf{p},\ \mathbf{q},\ \mathbf{r}$ be non-collinear. If $\mathbf{m_3}$ is an arbitrary point in the plane, we denote $\triangle_1 = \triangle_{\mathbf{m_1qr}}$, $\triangle_2 = \triangle_{\mathbf{m_2rp}}$, $\triangle_3 = \triangle_{\mathbf{m_3pq}}$ and $\triangle = \triangle_{\mathbf{pqr}}$. Using the second shape theorem for the triangle $\mathbf{pqr}$ and the points $\mathbf{m_1}$, $\mathbf{m_2}$, $\mathbf{m_3}$, we calculate

$$
\triangle_{\mathbf{m_1m_2m_3}} = \frac{\triangle\triangle_1 - (1-\triangle_1\triangle_3)(1-\triangle_3)^{-1}}{\triangle(1-\triangle_1\triangle_2)(1-\triangle_2)^{-1} - 1} = \frac{a\triangle_3 + b}{c\triangle_3 + d},
$$

where $a = (1-\triangle)\triangle_1(1-\triangle_2)$, $b = (\triangle\triangle_1 - 1)(1-\triangle_2)$,
$c = (1-\triangle_2) - (1-\triangle_1\triangle_2)\triangle$ and $d = (1-\triangle_1\triangle_2)\triangle - (1-\triangle_2)$.
From $\mathbf{m_1} \neq \mathbf{m_2}$, it follows that

$$
\triangle(1-\triangle_1\triangle_2)(1-\triangle_2)^{-1} - 1 \neq 0.
$$

Then, $c = -d \neq 0$ and

$$
ad - bc = (1-\triangle_1)(1-\triangle_2)^2 - (1-\triangle_1)(1-\triangle_2)(1-\triangle_1\triangle_2)\triangle \neq 0.
$$

Setting $\triangle_{\mathbf{m_3pq}} = \triangle_3 = z$ and $\triangle_{\mathbf{m_1m_2m_3}} = z'$, we obtain a bijective transformation ψ of the extended plane $\widetilde{\mathcal{E}}^2$ onto itself defined by

$$
\mathbf{C} \cup \infty \ \ni\ z = \triangle_{\mathbf{m_3pq}} \stackrel{\psi}{\longrightarrow} \triangle_{\mathbf{m_1m_2m_3}} = \frac{az+b}{cz+d} = z' \ \in\ \mathbf{C} \cup \infty.
$$

It is well - known that this linear - fractional transformation ψ is conformal. Moreover, ψ is not a similarity because $c = -d \neq 0$.

4. Bijective mappings of the shape sphere onto itself. Using the mapping σ defined in Section 2, we can replace the extended plane by the shape sphere. Thus, we construct some special bijections of the shape sphere.

Let $\varphi : \mathbf{C} \cup \infty \longrightarrow \mathbf{C} \cup \infty$ be the transformation defined by the equations (3) for $z \neq \infty$ and $\infty \stackrel{\varphi}{\longrightarrow} \infty$. Then we consider the composition $\Phi = \sigma \circ \varphi \circ \sigma^{-1}$ which is a one-to-one mapping of the shape sphere S^2 onto itself.

Theorem 1. *The transformation $\Phi : S^2 \longrightarrow S^2$ is a conformal mapping preserving the point C_1. Moreover, Φ is defined in the set $S^2 \setminus C_1$ by the equations*

$$
\begin{array}{rcl} u_1' & = & \dfrac{-f^2(u_1,u_2,u_3) - g^2(u_1,u_2,u_3) + h^2(u_1)}{2\{f^2(u_1,u_2,u_3) + g^2(u_1,u_2,u_3) + h^2(u_1)\}} \\ u_2' & = & \dfrac{f(u_1,u_2,u_3)h(u_1)}{f^2(u_1,u_2,u_3) + g^2(u_1,u_2,u_3) + h^2(u_1)} \\ u_3' & = & \dfrac{g(u_1,u_2,u_3)h(u_1)}{f^2(u_1,u_2,u_3) + g^2(u_1,u_2,u_3) + h^2(u_1)}, \end{array} \tag{4}
$$

where $S^2 \setminus C_1 \ni U = (u_1,u_2,u_3) \stackrel{\Phi}{\longrightarrow} (u_1',u_2',u_3') = U' \in S^2 \setminus C_1$ and

$$
\begin{array}{rcl} f(u_1,u_2,u_3) & = & (1 - a_1 - 2b_1)(1+2u_1) + 2\sqrt{3}(a_1u_2 + a_2u_3), \\ g(u_1,u_2,u_3) & = & (a_2 + 2b_2)(1+2u_1) - 2\sqrt{3}(a_2u_2 - a_1u_3), \\ h(u_1) & = & \sqrt{3}(1+2u_1). \end{array}
$$

Proof. Both mappings σ and σ^{-1} can be represented as a product of a stereographic projection (or its reverse mapping) and a linear-fractional mapping. Hence, the transformation Φ is conformal as a product of three conformal mappings σ^{-1}, φ and σ. From the equations (1) and (2), we have

$$S^2 \ni C_1 \xrightarrow{\sigma^{-1}} \infty \xrightarrow{\varphi} \infty \xrightarrow{\sigma} C_1.$$

Let $U = (u_1, u_2, u_3) \in S^2 \setminus C_1$. Then

$$\sigma^{-1}(U) = z = x + y.i = \frac{1 + 2u_1 - 2\sqrt{3}u_2}{2(1 + 2u_1)} + \frac{\sqrt{3}u_3}{1 + 2u_1}i.$$

Using (3), we get $\varphi(z) = z' = x' + y'.i$, where

$$\begin{aligned} x' &= \frac{a_1(1 + 2u_1 - 2\sqrt{3}u_2) - 2\sqrt{3}a_2u_3 + 2b_1(1 + 2u_1)}{2(1 + 2u_1)} \\ y' &= \frac{a_2(1 + 2u_1 - 2\sqrt{3}u_2) + 2\sqrt{3}a_1u_3 + 2b_2(1 + 2u_1)}{2(1 + 2u_1)}. \end{aligned} \tag{5}$$

Finally, the image of $z' \in \widetilde{\mathcal{E}}^2$ under the mapping σ is the point $U' \in S^2 \setminus C_1$ which coordinates are

$$\begin{aligned} u'_1 &= \frac{2x' - 2x'^2 - 2y'^2 + 1}{4(x'^2 + y'^2 + 1 - x')} \\ u'_2 &= \frac{\sqrt{3}(1 - 2x')}{4(x'^2 + y'^2 + 1 - x')} \\ u'_3 &= \frac{2\sqrt{3}y'}{4(x'^2 + y'^2 + 1 - x')}. \end{aligned}$$

Substituting x' and y' by the right-hand sides of (5), we obtain (4). This completes the proof.

The same approach can be used for the mapping $\psi : \widetilde{\mathcal{E}}^2 \longrightarrow \widetilde{\mathcal{E}}^2$. Then the composition $\Psi = \sigma \circ \psi \circ \sigma^{-1}$ also is a bijection of the shape sphere. Since ψ is a linear-fractional transformation different from a similarity, the bijection $\Psi : S^2 \longrightarrow S^2$ is a conformal transformation which does not preserve the point C_1.

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ПРЕОБРАЗУВАНИЯ НА ШЕЙП СФЕРАТА

Георги Христов Георгиев, Радостина Петрова Енчева

Две представяния на класовете от подобни триъгълници са разширената равнина и шейп сферата. Използвайки шейп теоремите за триъгълник, конструираме някои биективни изображения върху шейп сферата запазващи първата точка на съвпадане и получаваме в явен вид уравненията на тези изображения.