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ON THE NUMBER OF QUADRICS IN $\mathbb{R}^{n*}$

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In this note we give a short proof of the fact that the number of equivalent quadrics in $\mathbb{R}^n$ relative to affine transformations is $n^2 + 3n - 1$. The well known result for the number of equivalent quadrics relative to projective transformations is also considered. A short analysis of quadrics in $\mathbb{R}^n$, including the above results, may be an instructive complement to the standard material in the course of *Analytic Geometry* taught in the Technical and Economics Universities.

1. Introduction. When teaching the course of Analytic Geometry for non-mathematicians usually the main focus is put on the internal geometry of curves and surfaces and less attention is paid to the problem of classification (and counting) of these geometric figures. At the same time classification problems are traditionally of large interest in mathematics. Moreover, these problems for quadrics (figures described by quadratic equations) in $\mathbb{R}^2$, $\mathbb{R}^3$ and even in $\mathbb{R}^n$ are not difficult and may be well understood by students in engineering and economics. In this note we briefly discuss the problems of affine and projective classification of quadrics in $\mathbb{R}^n$ in a form suitable for teaching purposes, see e.g. [2].

2. The problem. A *quadric* in $\mathbb{R}^n$ is the set

$$Q = \{x \in \mathbb{R}^n : q(x) := x^\top Ax + 2b^\top x + c = 0\},$$

where $x = [x_1, \ldots, x_n]^\top$, $A \in \mathbb{R}^{n\times n}$ is a symmetric non-zero matrix, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $^\top$ denotes transposition. Two examples are the ellipsoid $\dfrac{x_1^2}{a_1^2} + \cdots + \dfrac{x_n^2}{a_n^2} - 1 = 0$ and the cone $\dfrac{x_1^2}{a_1^2} + \cdots + \dfrac{x_{n-1}^2}{a_{n-1}^2} - \dfrac{x_n^2}{a_n^2} = 0$.

Let $G \subset \mathbb{R}^{n\times n}$ be a multiplicative group of non-singular matrices. Consider the group Γ, acting on $\mathbb{R}^n$ as $x \mapsto \gamma(x) = Ux + x_0$, where $U \in \Gamma$ and $x_0 \in \mathbb{R}^n$. Two quadrics $P, Q \subset \mathbb{R}^n$ are said to be Γ*–equivalent* if there exist $\gamma \in \Gamma$ and $0 \neq \alpha \in \mathbb{R}$ such that

$$P = \{x \in \mathbb{R}^n : \alpha q(\gamma(x)) = 0\}.$$

The set of all quadrics, equivalent to a given quadric Q, is said to be the *orbit* of Q. Two orbits either coincide or are disjoint.

The interesting cases are when G is the group $\mathcal{O}(n)$ of real orthogonal $n \times n$ matrices or the group $\mathcal{GL}(n)$ of real non-singular $n \times n$ matrices.

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In the case $G = \mathcal{O}(n)$ the internal geometry of the quadric is preserved but there are "too many" (a continuum of) orbits. In particular two circles with radiuses $r_1 > 0$ and $r_2 > 0$ are in the same orbit if and only if $r_1 = r_2$. If we want all circles (or more generally, all ellipsoids) to be in the same orbit then we have to consider the group of affine transformations with $G = \mathcal{GL}(n)$.

Below we give a short proof [2] to the fact that the number a_n of different orbits of quadrics in $\mathbb{R}^n$ with respect to $G = \mathcal{GL}(n)$ is

$$a_n := n^2 + 3n - 1.$$

It must be pointed out that this result is not popular in the literature although it is well known that $a_1 = 3$, $a_2 = 9$ and $a_3 = 17$, see e.g. [1].

Remark. In this statement of the problem two quadrics may belong to different orbits when they define the empty set in $\mathbb{R}^n$. For example, the equations $x_1^2 + 1 = 0$ and $x_1^2 + x_2^2 + 1 = 0$ define non-equivalent quadrics in $\mathbb{R}^n$ for $n \geq 2$. Of course, considered as subsets of $\mathbb{C}^n$, the last two quadrics are non-equivalent non-empty figures.

A similar problem arises in the projective classification of quadrics. Set $x = u/v$ and $\xi = [\xi_1, \ldots, \xi_{n+1}]^\top := [u^\top, v]^\top \in \mathbb{R}^{n+1}$, where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$. Then the equation of the quadric Q in the projective space $\mathbb{P}^n$ becomes $\xi^\top \mathbf{A} \xi = 0$, where $\mathbf{A} := \begin{bmatrix} A & b \\ b^\top & c \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$. Here the admissible projective transformations are $\xi \mapsto \mathbf{U}\xi$, $\mathbf{U} \in \mathcal{GL}(n+1)$, and the problem is to determine the number p_n of orbits relative to this action of $\mathcal{GL}(n+1)$. The result $p_n = (n^2+6n+4)/4$ for n even and $p_n = (n^2+6n+5)/4$ for n odd is well known [1].

3. The solution. We may assume that the matrices A and $\mathbf{A}$ are diagonal with diagonal entries equal to 1, 0, or -1 since this can be achieved by suitable transformations $A \mapsto U^\top A U$ and $\mathbf{A} \mapsto \mathbf{U}^\top \mathbf{A} \mathbf{U}$ with $U \in \mathcal{GL}(n)$ and $\mathbf{U} \in \mathcal{GL}(n+1)$ respectively.

3.1. The affine case. In the elliptic case the matrix A is sign-definite and we may assume that it is equal to the identity matrix. Making a transformation $x \mapsto \lambda x - b$, $\lambda \in \mathbb{R}$, we obtain 3 quadrics: $x_1^2 + \cdots + x_n^2 - 1 = 0$ (an ellipsoid), $x_1^2 + \cdots + x_n^2 = 0$ (a point) and $x_1^2 + \cdots + x_n^2 + 1 = 0$ (the empty set in $\mathbb{R}^n$ or an imaginary ellipsoid in $\mathbb{C}^n$).

Consider next the hyperbolic case when $n \geq 2$ and the matrix A is non-singular and sign-indefinite. Then A may have m eigenvalues of one sign (say equal to 1 or -1) and $n - m$ eigenvalues of opposite sign (say equal to -1 or 1). Hence there are $n - 1$ hyperboloids

$$x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_n^2 - 1 = 0, \ m = 1, \ldots, n-1,$$

and $\left[\dfrac{n}{2}\right]$ cones

$$x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_n^2 = 0, \ m = 1, \ldots, n-1.$$

Here $[r]$ is the entire part of $r > 0$. In particular $\left[\dfrac{n}{2}\right] = n/2$ when n is even and $\left[\dfrac{n}{2}\right] = (n-1)/2$ when n is odd.

In the parabolic case when $n \geq 2$ the matrix A is singular with at least one zero eigenvalue. We have the following possibilities:

– one elliptic paraboloid $x_1^2 + \cdots + x_{n-1}^2 - x_n = 0$,

– or $\left[\dfrac{n-1}{2}\right]$ hyperbolic paraboloids

$$x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{n-1}^2 - x_n = 0,\ m = 1, \ldots, n-1,$$

– or a_{n-1} cylinders (we recall that a cylinder is a figure described by an equation of the type $f(x_{i_1}, \ldots, x_{i_s}) = 0$ with $s < n$).

Noting that $\left[\dfrac{n}{2}\right] + \left[\dfrac{n-1}{2}\right] = n-1$ we have

$$a_n = 3 + \left(n - 1 + \left[\frac{n}{2}\right]\right) + \left(1 + \left[\frac{n-1}{2}\right] + a_{n-1}\right) = a_{n-1} + 2n + 2.$$

The solution of this difference equation in a_n is

$$a_n = a_1 + \sum_{k=2}^{n} (2k+2).$$

Having in mind that $a_1 = 3$ we obtain

$$a_n = n^2 + 3n - 1.$$

3.2. The projective case. The number p_n of different orbits of quadrics in $\mathbb{R}^n$ relative to the projective equivalence relation satisfies the difference equation

$$p_n = 1 + \left[\frac{n+1}{2}\right] + p_{n-1},\ n \geq 2.$$

Indeed, we have one quadric $\xi_1^2 + \cdots + \xi_{n+1}^2 = 0$, or $\left[\dfrac{n+1}{2}\right]$ quadrics

$$\xi_1^2 + \cdots + \xi_m^2 - \xi_{m+1}^2 - \cdots - \xi_{n+1}^2 = 0,\ m = 1, \ldots, \left[\frac{n+1}{2}\right].$$

Since $p_1 = 3$ we get

$$p_n = 3 + \sum_{k=2}^{n} \left(1 + \left[\frac{k+1}{2}\right]\right) = \frac{2n^2 + 12n + 9 + (-1)^{n+1}}{8},\ n \geq 2.$$

4. Quadrics in low dimensions. When teaching the elements of multi-dimensional analytic geometry it is instructive to illustrate the theory by low-dimensional examples.

4.1. The case $n = 1$. We have only the elliptic class with representatives $x_1^2 - 1 = 0$ (two points), $x_1^2 = 0$ (a double point) and $x_1^2 + 1 = 0$ (the empty set). Hence $a_1 = 3$.

There are also $p_1 = 3$ figures according to the projective classification, namely $\xi_1^2 + \xi_2^2 = 0$, $\xi_1^2 - \xi_2^2 = 0$ and $\xi_1^2 = 0$.

4.2. The case $n = 2$. The elliptic case involves an ellipse $x_1^2 + x_2^2 - 1 = 0$, a point $x_1^2 + x_2^2 = 0$ and the empty set $x_1^2 + x_2^2 + 1 = 0$.

The hyperbolic case contains a hyperbola $x_1^2 - x_2^2 - 1 = 0$ and a cone (a pair of intersecting straight lines in this case) $x_1^2 - x_2^2 = 0$.

The parabolic case includes a parabola $x_1^2 - x_2 = 0$ plus $p_1 = 3$ cylinders (straight lines in this case)

The total number of figures is 9 which is also the number $a_2 = 2^2 + 3 \cdot 2 - 1 = 9$.

According to the projective classification we have 2 quadrics $\xi_1^2 + \xi_2^2 + \xi_3^2 = 0$ and $\xi_1^2 + \xi_2^2 - \xi_3^2 = 0$ plus $p_1 = 3$ cylinders. Thus $p_2 = 5$.

4.3. The case $n = 3$. The elliptic case contains an ellipsoid $x_1^2 + x_2^2 + x_3^2 - 1 = 0$, a point $x_1^2 + x_2^2 + x_3^2 = 0$ and the empty set $x_1^2 + x_2^2 + x_3^2 + 1 = 0$.

The hyperbolic case involves 2 hyperboloids $x_1^2 - x_2^2 - x_3^2 - 1 = 0$, $x_1^2 + x_2^2 - x_3^2 - 1 = 0$ and a cone $x_1^2 - x_2^2 - x_3^2 = 0$.

In the parabolic case there are 2 paraboloids $x_1^2 + x_2^2 - x_3 = 0$, $x_1^2 - x_2^2 - x_3 = 0$ and $a_2 = 9$ cylinders.

Thus the number of quadrics in this case is 17, which coincides with $a_3 = 3^2 + 3\cdot 3 - 1 = 17$.

In the projective case we have 3 quadrics $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 0$, $\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 = 0$, $\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 = 0$ plus $p_2 = 5$ cylinders, i.e., $p_3 = 8$.

4.4. The case $n = 4$. The elliptic case includes an ellipsoid $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1 = 0$, a point $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ and the empty set $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 1 = 0$.

The hyperbolic case contains 3 hyperboloids $x_1^2 - x_2^2 - x_3^2 - x_4^2 - 1 = 0$, $x_1^2 + x_2^2 - x_3^2 - x_4^2 - 1 = 0$, $x_1^2 + x_2^2 + x_3^2 - x_4^2 - 1 = 0$ and 2 cones $x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$, $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0$.

The parabolic case involves 2 paraboloids $x_1^2 + x_2^2 + x_3^2 - x_4 = 0$, $x_1^2 + x_2^2 - x_3^2 - x_4 = 0$ plus $a_3 = 17$ cylinders. Hence the number of quadrics here is 27, which corresponds to $a_4 = 4^2 + 3 \cdot 4 - 1 = 27$.

In the projective case we have 3 quadrics $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 = 0$, $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_5^2 = 0$, $\xi_1^2 + \xi_2^2 + \xi_3^2 - \xi_4^2 - \xi_5^2 = 0$ and $p_3 = 8$ cylinders, i.e., $p_4 = 11$.

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ВЪРХУ БРОЯ НА КВАДРИКИТЕ В $\mathbb{R}^n$

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Дадено е кратко доказателство на факта, че броят на еквивалентните квадрики в $\mathbb{R}^n$ относно афинните трансформации е $n^2 + 3n - 1$. Разгледан е и добре известният резултат относно броя на еквивалентните квадрики относно проективната класификация. По мнение на автора, един кратък класификационен анализ на квадриките в $\mathbb{R}^n$, включващ горните резултати, е добро допълнение към стандартния материал по дисциплината "Аналитична геометрия", преподавана в техническите и икономическите университети.