МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2002
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2002
Proceedings of Thirty First Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 3–6, 2002

LOWER ESTIMATOR OF A NUMBER OF POSITIVE INTEGERS h, FOR WHICH EXIST INFINITELY MANY PRIMES h-TWINS*

Tatiana Liubenova Todorova

In this paper the upper estimator for module of sum of complex functions of integer argument is obtained. The number of positive integers h, for which there exist infinitely many primes p such that $p + 2h$ is prime, is derived too.

1. Introduction. One of the ancient problems in number theory is the hypothesis for the number of twin primes. It states that exist infinitely many primes p, for which $p+2$ is prime, too. In 1919, W. Brun proved, that there are infinitely many natural numbers n, such that n and $n+2$ possess not more than 9 prime divisors. The positive integers p and $p+2h$ are called h-twins, if they are simultaneously primes. In 1923 G. Hardy and J. Litlewood introduced the hypothesis, that number of primes $p \le N, N \in \mathbb{N}$, such that $p + 2h$ $(h \in \mathbb{N})$ is prime, is given by the formula:

$$Z(N, 2h) \sim 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{2<p/h} \frac{p-1}{p-2} \cdot \frac{N}{\ln^2 N},$$

where: $Z(N, 2h) = Card\{p \in \mathbb{N} : p \le N, \quad p, p+2h - \text{primes}\}$; "$p/h$" denotes that p divides h; if $h = 2^k, k \in \mathbb{N}$ the second product is equal to 1. In 1961, A. Lavrik [3] proved that the equality

$$Z(N, 2h) = 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{2<p/h} \frac{p-1}{p-2} \cdot \frac{N}{\ln^2 N} + O\left(\frac{N}{\ln^3 N}\right) \qquad (1)$$

is fulfilled for each $h \in \mathbb{N} \cap \left(0, \dfrac{N}{2 \ln N}\right)$ except not more than $\dfrac{\gamma N}{(\ln N)^M}$, $\gamma \in \mathbb{R}^+ \setminus \{0\}$ numbers; where: M is a constant, $M \in (1, \infty)$; the number γ and remainder term in equality (1) depend only on M. The constraint $h \in \mathbb{N} \cap \left(0, \dfrac{N}{2 \ln N}\right)$ is due to the fact there not enough good estimate for the number of primes in arithmetic progressions with

*2000 Math. Subject Classification: 11A41, 11N05, 11N80 difference greater than $\dfrac{N}{\ln N}$. Using sieve methods H. Halberstam, H. Richert [1] (Th 3.11, ch. 3, § 7, [1]) received for $Z(N,2h)$ the following estimator:

$$Z(N,2h) \le 8 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \prod_{2<p/h} \frac{p-1}{p-2} \cdot \frac{N}{\ln^2 N},$$

(2)

$$\prod_{p>2} \frac{p(p-2)}{(p-1)^2} \le 0.6601\ldots; \quad \prod_{2<p/h} \frac{p-1}{p-2} \le \ln\ln\ln(3h) + O(1),$$

which is the best one till now.

We shall use the following notations:

$m_1, m_2, k, h, N \in \mathbb{N}$ – positive integers;

p – prime number;

$\pi(N) = \sum_{p \le N} 1$ – the number of primes $p \le N$;

$[a]$ – the integer part of a, resulting in the greatest integer less or equal to a;

$\lambda(n)$ – is Mangold's function:

$$\lambda(n) = \begin{cases} \ln p, & \text{for } n = p^k \\ 0, & \text{for } n \ne p^k. \end{cases}$$

$Z(N,2h) = \sum_{\substack{p \le N \\ p,p+2h-\text{primes}}} 1 = Card\{p \in \mathbb{N} : p \le N, p, p+2h - \text{primes}\}$ – number of primes p, such that $p+2h$ is a prime, too.

The asymptotic law of primes distribution $\pi(N) \sim \dfrac{N}{\ln N}$ is applied, too.

2. Main results. We shall prove the following generalization of Lemma 3, ch. 1, § 3, [2]:

Theorem 1. *Let the following conditions hold:*

1. $\varphi : (a,b] \to \mathbb{C}$ *is a complex function;*
2. $q \le [b] - [a]$, $q \in \mathbb{N}$.

Then

$$\left| \sum_{a<n\le b} \varphi(n) \right| \le \frac{[b]-[a]}{\sqrt{q}} \sup_{a<n\le b} |\varphi(n)| + \sqrt{\frac{2([b]-[a])}{q} \sum_{r=1}^{q-1} \left| \sum_{a<n\le b-r} \varphi(n)\overline{\varphi}(n+r) \right|}.$$

Proof. We set $\varphi(n) = 0$ for $n \in \mathbb{Z} \setminus (a,b]$. Then

$$S = \sum_{a<n\le b} \varphi(n) = \sum_n \varphi(n+m), \text{ for } m \in \mathbb{Z}.$$

In this sum we have finitely many summands, so we can write:

$$S = \frac{1}{q} \sum_n \sum_{m=1}^{q} \varphi(n+m).$$

Using Cauchy's inequality, we obtain:

$$|S|^2 \le \frac{1}{q^2}\Big(\sum_n 1\Big)\cdot\Big(\sum_n \Big|\sum_{m=1}^{q}\varphi(n+m)\Big|^2\Big).$$

From the fact that $\varphi(n)\neq 0$ for $[b]-[a]$ numbers it follows

$$|S|^2 \le \frac{[b]-[a]}{q^2}\sum_n \Big|\sum_{m=1}^{q}\varphi(n+m)\Big|^2 \le$$

$$\le \frac{[b]-[a]}{q^2}\left\{\sum_n\sum_{m=1}^{q}|\varphi(m+n)|^2 + 2\left|\sum_n\sum_{1\le m<s\le q}\varphi(n+m)\overline{\varphi}(n+s)\right|\right\} \le$$

(3) $$\le \frac{[b]-[a]}{q^2}\left\{\sum_{m=1}^{q}\sum_{n=1}^{N-m}|\varphi(m+n)|^2 + 2q\sum_{r=1}^{q}\left|\sum_{n=1}^{N-r}\varphi(n)\overline{\varphi}(n+r)\right|\right\}$$

Further the proof of theorem is similar to the proof of Lemma 3, ch. 1, § 3, [2]. □

For $\varphi(n)=e^{2\pi i f(n)}$ we obtain the statement of quote above Lemma:

Corollary 1. *Let the following conditions be fulfilled:*

1. $f:(a,b]\to\mathbb{R}$ *is a real function;*

2. $g(n)=f(n+r)-f(n)$, $n\le(a,b]\cap\mathbb{N}$, $r\in[1,q-1]\cap\mathbb{N}$, $q\in[2,[b]-[a]]\cap\mathbb{N}$*;*

Then

$$\left|\sum_{a<n\le b}e^{2\pi i f(n)}\right| \le \frac{[b]-[a]}{\sqrt{q}} + \sqrt{\frac{2([b]-[a])}{q}\sum_{r=1}^{q-1}\left|\sum_{a<n\le b-r}e^{2\pi i g(n)}\right|}.$$

Corollary 2. *Let the following conditions be valid:*

1. $N\in\mathbb{N}$*;*

2. $\varepsilon\in\left(\frac{1}{2},1\right)$*;*

3. $h\in\mathbb{N}\cap\left(0,\frac{N}{2}\right)$.

Then there exist at least $\left[\frac{(2\varepsilon-1)N}{11.2\ln\ln\ln(3N)}\right]$ *numbers* h*, such that* $Z(N,2h)\ge\frac{(1-\varepsilon)N}{\ln^2 N}$.

Proof. We consider the sum $S=\sum_{n=1}^{N}\frac{\lambda(n)}{\ln n}$. The following equality is valid:

$$\begin{aligned} S &= \sum_{1<n\le N}\frac{\lambda(n)}{\ln n} = \sum_{p\le N}1 + \sum_{p\le\sqrt{N}}\frac{1}{2} + \sum_{p\le\sqrt[3]{N}}\frac{1}{3} + \cdots = \\ &= \pi(N) + \frac{1}{2}\pi(\sqrt{N}) + \frac{1}{3}\pi(\sqrt[3]{N}) + \cdots \end{aligned}$$

Evidently

$$S \geq \pi(N). \tag{4}$$

Let $q \in [1, N] \cap \mathbb{N}$. For $\varphi(n) = \dfrac{\lambda(n)}{\ln n}$ applying inequality (3) one obtains the estimate:

$$|S|^2 \leq \frac{N}{q^2}\left\{\sum_{m=1}^{q}\sum_{n=1}^{N-m}\frac{\lambda^2(n+m)}{\ln^2(n+m)} + 2q\sum_{r=1}^{q}\sum_{n=1}^{N-r}\frac{\lambda(n)\lambda(n+r)}{\ln n \ln(n+r)}\right\}.$$

From the above inequality and (4) we have:

$$\Big(\pi(N)\Big)^2 \leq \frac{N}{q^2}\Big(S_1 + 2qS_2\Big), \tag{5}$$

$$\text{where } S_1 = \sum_{m=1}^{q}\sum_{n=1}^{N-m}\frac{\lambda^2(n+m)}{\ln^2(n+m)},\ S_2 = \sum_{r=1}^{q}\sum_{n=1}^{N-r}\frac{\lambda(n)\lambda(n+r)}{\ln n \ln(n+r)}.$$

We shall evaluate the sums S_1 and S_2.
The following inequality is valid:

$$S_1 \leq \sum_{m=1}^{q}\sum_{n=1}^{N}\left(\frac{\lambda(n)}{\ln n}\right)^2 = \sum_{m=1}^{q}\left(\pi(N) + \frac{1}{2^2}\pi(\sqrt{N}) + \frac{1}{3^2}\pi(\sqrt[3]{N}) + \cdots\right) \leq$$

$$\leq q\pi(N) + q\pi(\sqrt{N})\left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \leq q\pi(N) + 2q\pi(\sqrt{N}). \tag{6}$$

The definition of Mangold's function yields $\lambda(n) \neq 0$ for $n = p^k, p \geq 2$. Then:

$$S_2 = S_{21} + S_{22},\ \text{where} \tag{7}$$

$$S_{21} = \sum_{r=1}^{q}\sum_{n=2^k \leq N-r}\frac{\lambda(n)\lambda(n+r)}{\ln n \ln(n+r)},\ S_{22} = \sum_{r=1}^{[\frac{q-1}{2}]}\sum_{\substack{n=p^k \leq N-2r \\ p>2}}\frac{\lambda(n)\lambda(n+2r)}{\ln n \ln(n+2r)}.$$

The number of integers $n = 2^k < N$ is $[\log_2 N]$. In accordance with Euler's constant $C = \lim\limits_{n\to\infty}\left(\sum_{\nu=1}^{n}\dfrac{1}{\nu} - \ln n\right) = 0.5772\ldots$, we obtain for S_{21} the following inequality:

$$S_{21} \leq \sum_{r=1}^{q}\sum_{k \leq [\log_2 N]}\frac{1}{k} \leq 2q \ln \log_2 N \leq 4q \ln\ln N. \tag{8}$$

We are going to show that:

$$\sum_{\substack{n=p^k \leq N-2r \\ p>2}}\frac{\lambda(n)\lambda(n+2r)}{\ln n \ln(n+2r)} \leq Z(N-2r, 2r) + 2\ln N. \tag{9}$$

For $p > 2$ and $p, p+2r$ – simultaneously primes the following equality is valid:

$$\sum_{\substack{n=p^k\le N-2r\\ p>2}} \frac{\lambda(n)\lambda(n+2r)}{\ln n\ln(n+2r)} = \sum_{p\le N-2r} 1 + \sum_{p^2, p+2r\le N} \frac{1}{2} +$$

$$+ \sum_{p,(p+2r)^2\le N} \frac{1}{2} + \sum_{p^3, p+2r\le N} \frac{1}{3} + \sum_{p,(p+2r)^3\le N} \frac{1}{3} +$$

$$+ \sum_{p^2,(p+2r)^2\le N} \frac{1}{4} + \sum_{p^3,(p+2r)^2\le N} \frac{1}{6} + \sum_{p^2,(p+2r)^3\le N} \frac{1}{6} + \cdots \tag{10}$$

The greatest k, for which $p^k \le N$, $2 < p \le N$ is $[\log_3 N]$. There are $k-1$ summands corresponding to any power k, $k \in [2, [\log_3 N]]$ of the following form:

$$\sum_{p^{m_1},(p+2r)^{m_2}\le N} \frac{1}{m_1 m_2},$$

where $m_1 + m_2 = k$; $m_1, m_2 \in [1, k) \cap \mathbb{N}$. For $k \ge 3$ we have:

$$\sum_{k=3}^{[\log_3 N]} \Bigg(\sum_{\substack{p^{m_1},(p+2r)^{m_2}\le N\\ m_1+m_2=k}} \frac{1}{m_1 m_2} \Bigg) \le \sum_{k=3}^{[\log_3 N]} \frac{k-1}{k-1} < 2\ln N. \tag{11}$$

Applying the definition of function $Z(N, 2h)$, (10) and (11) we obtain the inequality (9). Hence for S_{22} we derive the following estimator:

$$S_{22} \le q\ln N + \sum_{r=1}^{[\frac{q-1}{2}]} Z(N-2r, 2r).$$

The above inequality and (8) yield:

$$S_2 \le 4q\ln\ln N + q\ln N + \sum_{1\le r\le[\frac{q-1}{2}]} Z(N-2r, 2r). \tag{12}$$

Using (5), (6) and (12) we obtain:

$$\Big(\pi(N)\Big)^2 \le \frac{N}{q^2}\Big\{ q\pi(N) + 2q\pi(\sqrt{N}) + 8q\ln\ln N +$$

$$+2q^2\ln N + 2q \sum_{1\le r\le[\frac{q-1}{2}]} Z(N-2r, 2r)\Big\}. \tag{13}$$

Let s be a number of positive integers $2r$, $2r < N$, such that $Z(N-2r, 2r) \ge \dfrac{(1-\varepsilon)N}{\ln^2 N}$. In accordance with (2) we have:

$$\max_{1\le r\le[\frac{N-1}{2}]} Z(N-2r, 2r) \le$$

$$\leq 5.6 \sup_{1\leq r\leq[\frac{N-1}{2}]} \frac{N-2r}{\ln^2(N-2r)} \max\{2, \sup_{3<r\leq[\frac{N-1}{2}]} \ln\ln\ln(3r)\} \leq A(N)\frac{N}{\ln^2 N},$$

where $A(N) < 5.6 \ln\ln\ln(3N)$.

Therefore, it follows from the inequality (13) that:

$$\left(\pi(N)\right)^2 \leq \frac{N}{q}\pi(N) + \frac{2(q-s)N}{q}.\frac{(1-\varepsilon)N}{\ln^2 N} + \frac{2N}{q}A\frac{N}{\ln^2 N}. \tag{14}$$

From the fact that $\lim\limits_{N\to\infty} \dfrac{\pi(N)\ln N}{N} = 1$, dividing both sides of (14) by $\dfrac{N^2}{\ln^2 N}$ and performing the limiting conversion as $N \to \infty$ we have:

$$1 \leq \lim_{N\to\infty} \frac{\ln N}{q} + \lim_{N\to\infty} 2\left(1-\frac{s}{q}\right)(1-\varepsilon) + \lim_{N\to\infty} 2A\frac{s}{q}.$$

Let $q = N$. Then for enough large N the following inequality is valid:

$$s \geq \frac{(2\varepsilon-1)N}{2A} \geq \frac{(2\varepsilon-1)N}{11.2\ln\ln\ln(3N)}.$$

The choice of number s and the above inequality yield the statement of Corollary 2.

REFERENCES

[1] H. Nalberstam, H. Richert. Sieve Methods, London, Academic Press Inc, 1974.

[2] А. А. Карацуба. Основы аналитической теории чисел, Москва, Наука, 1983.

[3] А. Ф. Лаврик. К бинарным проблемам аддитивной теории простых чисел в связи с методом тригонометрических сумм И. М. Виноградова. *Вестник Ленинградского Университета*, **13** (1961), 11–27.

Department of Algebra
Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5 J.Baurchier Blvd
1164 Sofia, Bulgaria
e-mail: tlt@fmi.uni-sofia.bg

ОЦЕНКА ОТДОЛУ НА БРОЯ НА ЧИСЛАТА h, ЗА КОИТО СЪЩЕСТВУВАТ БЕЗБРОЙ МНОГО ПРОСТИ ЧИСЛА h-БЛИЗНАЦИ

Татяна Любенова Тодорова

В статията е получена оценка отгоре за модула на сума от стойностите на комплексните функции от целочислен аргумент. Намерена е оценка отдолу за броя на естествените числа h, за които съществуват безброй много прости числа p, такива че $p + 2h$ също е просто.