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AN OUTSIDE VIEW ON THE JORDAN CURVE THEOREM

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A compact set K in $\mathbf{R}^n$ is said to be an irreducible boundary if $\mathbf{R}^n \setminus K$ is disconnected and K is a boundary of each component of $\mathbf{R}^n \setminus K$. This paper contains a necessary and sufficient condition which describes irreducible boundaries in $\mathbf{R}^n$: the compactum $K \subset \mathbf{R}^n$ is an irreducible boundary if and only if it is a frame of some minimal essential family. As a corollary the classical Jordan curve theorem is obtained.

1. Introduction. The classical Jordan Curve Theorem states the fact which is geometrically quite "obvious": every simple closed curve in the plane $\mathbf{R}^2$ divides the plane into two pieces, the "inside" and "outside" of the curve, and it is their common boundary.

Speaking more generally the Jordan theorem states that if K is a homeomorphic image of the $(n-1)$-dimensional unit sphere S^{n-1} in $\mathbf{R}^n$ then K separates $\mathbf{R}^n$ into two domains (connected open sets) and K is their common boundary. In other words K is an irreducible boundary in $\mathbf{R}^n$.

Definition 1.1. *The compactum K in the n-dimensional space $\mathbf{R}^n$ is said to be an irreducible boundary if and only if $\mathbf{R}^n \setminus K$ is disconnected and K is a boundary of each component of $\mathbf{R}^n \setminus K$.*

Note that for any proper closed subset H of K the above assertion fails: $\mathbf{R}^n \setminus H$ is a connected set.

Certainly the topological images of the sphere are not at all the only examples of irreducible boundaries in $\mathbf{R}^n$; we may point here for the example "Wada lakes" or more simple constructions like the so called "Warsaw circumference": $W = \{(0,y) | -2 \le y \le 1\} \cup \{(x,-2) | 0 \le x \le 1\} \cup \{(1,y) | -2 \le y \le \sin 1\} \cup \{(x,y) | y = \sin \frac{1}{x}; x \in (0,1]\} \subset \mathbf{R}^2$ and similar surfaces in the space $\mathbf{R}^n$.

In this paper we give a common description of all irreducible boundaries in $\mathbf{R}^n$. Our considerations utilize classical topology technique and are based on one of the fundamental (and also classical) notions in dimension theory: essential families. We also reprove here some fundamental facts of the dimension theory in (probably) more easy way.

We recall below some standard denotations and well known definitions.

All topological spaces below are subsets of the Euclidean n-dimensional space $\mathbf{R}^n$ with the standard norm.

For $X \subset \mathbf{R}^n$ we denote by $O_\varrho(X) = \{x \in \mathbf{R}^n | d(x,X) < \varrho\}$ and $B_\varrho(X) = \{x \in \mathbf{R}^n | d(x,A) \le \varrho\}$ the open and closed ϱ-neighbourhoods of the set $X \subset \mathbf{R}^n$. Thus $O_\varrho(x)$ and $B_\varrho(x)$ are open and closed balls respectively, centered in $x \in \mathbf{R}^n$ and with radius ϱ.

We also put $S_\varrho(x) = B_\varrho(x) \setminus O_\varrho(x)$ the $(n-1)$-dimensional sphere with a center at x and radius ϱ.

Definition 1.2. *The closed set C of a topological space X is said to be a partition (or separator) between F and G if $X \setminus C = U \cup V$ where U and V are open sets; $U \supset F$; $V \supset G$ and $U \cap V = \emptyset$.*

Definition 1.3. *Let $\Sigma = \{(F_{-1}, F_{+1}), \ldots, (F_{-n}, F_{+n})\}$ be a family of disjoint pairs of closed subsets of the topological space X. Σ is said to be an essential family (often shortened to n-family) if for an arbitrary collection $\{C_i\}_{i=1}^n$, of separators C_i in X between $\{F_{-i}\}$ and $\{F_{+i}\}$ we have $\bigcap\limits_{i=1}^{n} C_i \neq \emptyset$.*

One of the basic results in dimension theory states that for normal spaces X we have $\dim X \geq n$ if and only if there exists an essential family in X [4,5].

In describing the irreducible boundaries, we make use the following term.

Frame of an essential family – the set $|\Sigma| = \bigcup\limits_{i=1}^{n} (F_{-i} \cup F_{+i})$ is called a frame (or *fence*) of Σ.

Definfition 1.4. *An essential n-family Σ in a given topological space X is said to be minimal in X if for an arbitrary proper closed set $Y \subset X$ the restriction $\Sigma|_Y = \{(Y \cap F_{-i}, Y \cap F_{+i}\}_{i=1}^n$ is not essential family in Y.*

Definition 1.5. *The subspace K of a given topological space X is said to be a fence, if it is a minimal frame in X. This means that one can find such an essential family Σ in X that $K = |\Sigma|$ and if H is a proper subset of K, then the restriction $\Sigma|_H = \{H \cap F_{\pm i}\}_{i=1}^n$ is not essential in X.*

Further we refer to the space X as a *membrane* spanned on Σ if for any nonempty open set $U \subset X$, the family Σ is not essential in the space $(X \setminus U) \cup |\Sigma|$.

A classical example of an essential family is given by Lebesgue [1] – the collection of the opposite faces of an n-dimensinal cube I^n; $I = [-1, 1]$. Here $F_{\pm i} = \{x \in I^n | x_i = \pm 1\}$ (it is easy to prove by using the Sperner lemma). It is a trivial observation also that I^n is a membrane on its fence: for an arbitrary point $p \in Int(I^n)$ the separators $C_i = \{x \in I^n | x_i = p_i\}$ have as an intersection the single point set $\{p\}$.

The main result in the present paper states that the irreducible boundaries in $\mathbf{R}^n$ are quite similar to the fence of I^n:

Theorem: *The compact subset K of $\mathbf{R}^n$ is an irreducible boundary if and only if K is a fence in $\mathbf{R}^n$.*

2. Propositions and Lemmas.

Proposition 2.1 ([9]). *If for a compact set $K \subset \mathbf{R}^n$ the complement $\mathbf{R}^n \setminus K$ is disconnected, then K is a frame (not necessary a fence) of an essential family.*

Proof. Choose a bounded component U of $\mathbf{R}^n \setminus K$ and fix some point $p \in U$. Clearly one may take $a > 0$ large enough for which the inclusion $K \cup U \subset Q = [-a, a]^n$ holds. Denote by $Q_{\pm i}$ the opposite faces $Q_{\pm i} = \{x \in Q | x_i = \pm a\}$ of the n-dimensional cube Q and let $P_{\pm i}$ be the pyramid with a base $Q_{\pm i}$ and as a vertice the point p. We set $K_{\pm i} = K \cap P_{\pm i}$. It is easy to see that the desired essential family is $\Sigma = \{(K_{-i}, K_{+i})\}_{i=1}^n$.

Furthermore recall that for given $0 < \delta < \varrho$ and for $O = O_\varrho(p)$, $B = B_\delta(p)$ one can find homeomorphisms $g : O \to \mathbf{R}^n$ and $h : \mathbf{R}^n \setminus B \to \mathbf{R}^n \setminus \{p\}$ such that each of the restrictions $g|_B$ and $h|_{\mathbf{R}^n \setminus O}$ is the identity.

Lemma 2.2. *Suppose that $\Sigma = \{K_{\pm i}\}_{i=1}^n$ is minimal in the set $H \subset \mathbf{R}^n$ and let p be an inner point of H. Then there exists a family $\{C_i\}_{i=1}^n$ of separators between K_{-i} and K_{+i} in H for which $\bigcap\limits_{i=1}^{n} C_i = \{p\}$.*

Proof. Choose the positive numbers ε and ϱ so that $p \in B = B_\varepsilon(p) \subset O = O_\varrho \subset H$. The family Σ is not essential in the proper closed subset $H \setminus O_\varepsilon$ such that Σ is minimal. Hence one can find partitions C_i' between K_{-i} and K_{+i} with an empty intersection. By using standard topological techniques we can expand each C_i' to a separator $\widetilde{C}_i$ between K_{-i} and K_{+i} in H.

Apparently $\emptyset \neq \bigcap\limits_{i=1}^{n} \widetilde{C}_i \subset B$. Now we use the previous Lemma and take a homeomorphism $h : \mathbf{R}^n \setminus B \to \mathbf{R}^n \setminus \{p\}$. It is easy to see that $C_i = \{p\} \cup h(\widetilde{C}_i)$ is a separator between $h(K_{-i}) = K_{-i}$ and $h(K_{+i}) = K_{+i}$ in $h(H) = H$. It is clear that $\bigcap\limits_{i=1}^{n} C_i = \{p\}$.

Now let γ be a piecewise linear path which connects the points p and q in $\mathbf{R}^n$ and for $\varepsilon > 0$ $P = O_\varepsilon(\gamma)$ be the ε-neighbourhood of γ. It is a folklore fact that there exists a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ with $h(p) = q$ and which is the identity on $\mathbf{R}^n \setminus O$.

Proposition 2.3. *[9] Let $K \subset \mathbf{R}^n$ be a compact frame of the essential family Σ (i.e. $|\Sigma| = K$). Then $\mathbf{R}^n \setminus K$ is not connected.*

Hint. Let O and B be open and closed balls, centered for example at the origin 0 and $K \subset B \subset O$. As in the Lemma 2.3, choose a homeomorphism $g : O \to \mathbf{R}^n$ with $g|_B = id$. Note that $g(K) = K = g^{-1}(K)$, hence Σ is essential in O. Clearly it is essential in every space which contains O, in particular Σ is an essential family in the closure B of O.

Thus it is clear that we can divide the points of $\mathbf{R}^n \setminus K$ into two classes, say U and V: $p \in U$ if and only if Σ is still essential in $\mathbf{R}^n \setminus \{p\}$ and $V = \mathbf{R}^n \setminus (K \cup U)$. V is nonempty by means of Lemma 2.2 and $U \supset \mathbf{R}^n \setminus B \neq \emptyset$. According to the above remark there is no piecewise linear path which connects a point of U with a point of V which copletes the proof.

Lemma 2.4. *If the simplicial complex K is an irreducible boundary in $\mathbf{R}^n$ then K divides $\mathbf{R}^n$ into two connected components.*

Proof. Because each irreducible boundary in $\mathbf{R}^1$ is a point, we suppose $n \geq 2$. Let P be a simplex of K and let $L = \mathrm{span}(P)$. Clearly L is an $(n-1)$-dimensial hyperplane in $\mathbf{R}^n$ which is homeomorphic to $\mathbf{R}^{n-1}$. Let us pick a point $p \in Int_L(T)$. Consider the set $Q = p + \frac{1}{2}(P - p)$ – the image of P under the homothety with a factor $\frac{1}{2}$ and a ray center at p. Clearly Q and $K \setminus Int_L(P)$ are two disjoint compact sets so one can choose a positive $\varepsilon < d(Q, K \setminus Int_L(P))$. Choose furthermore a unit vector e in $\mathbf{R}^n$ which is orthogonal to L. The open set $U = Int_L(Q) \times (-\varepsilon, \varepsilon) = \bigcup\limits_{x \in Int_L(Q)} (x - \varepsilon e, x + \varepsilon e)$ meets K only in the points of $Int_L(Q)$ and every such point is a boundary point for exactly two components: $Int_L(Q) \times (-\varepsilon, 0)$ and $Int_L(Q) \times (0, \varepsilon)$.

3. The main Theorem and corollaries.

Proof of the Theorem of Section 1. Let $K \subset \mathbf{R}^n$ be an irreducible boundary. It follows by Proposition 2.1 that there exists an essential family $\Sigma = \{(K_{-i}, K_{+i})\}_{i=1}^n$ with $|\Sigma| = K$. Suppose that $H \subset K$ is closed and $K \setminus H \neq \emptyset$. If the family $\Sigma|_H$ was essential then according to Proposition 2.3 the set $\mathbf{R}^n \setminus H$ would be disconnected. But by our hypothesis K is an irreducible boundary, so the set $\mathbf{R}^n \setminus H$ is connected. Hence $\Sigma|_H$ is not essential and thus Σ is a fence.

Suppose now that $K = |\Sigma|$ is a fence of the minimal essential n-family Σ. Then using Proposition 2.3 we see that $\mathbf{R}^n \setminus K$ is not connected. Supposing that for some proper closed subset H of K the set $\mathbf{R}^n \setminus H$ remains disconnected, we obtain by Proposition 2.1 that the family $\Sigma|_H$ is still essential. This contradicts the assumption that Σ is minimal. Hence K is an irreducible boundary.

Corollary 3.1. *(The Jordan Separation Theorem), Let $f : S^{n-1} \to \mathbf{R}^n$ be an embedding of the $(n-1)$-dimensional sphere in the n-dimensional Euclidean space. Then $\mathbf{R}^n \setminus f(S^{n-1})$ is disconnected and $f(S^{n-1})$ is a boundary of each component of its complement.*

Proof. We consider S^{n-1} as a boundary of the n-dimensional cube I^n. As was mentioned above, S^{n-1} is the fence of the standard essential family $\Phi = \{(F_{-i}, F_{+i})\}_{i=1}^n$ in I^n. Put $K = f(S^{n-1})$ and suppose that $\Sigma = f(\Phi)$ is not essential ($\Sigma = \{(K_{-i}, K_{+i})\}_{i=1}^n$ where $K_{\pm i} = f(I_{\pm i})$. Then there exists a family $\{C_i\}_{i=1}^n$ between K_{-i} and K_{+i} in $\mathbf{R}^n$ with an empty intersection. We set $\mathbf{R}^n \setminus C_i = U_{-i} \cup U_{+i}$ where $U_{\pm i} \supset K_{\pm i}$; $U_{\pm i}$ are open and $U_{-i} \cap U_{+i} = \emptyset$.

Consider $\mathbf{R}^n$ as the hyperplane $\{x \in \mathbf{R}^{n+1} | x_{n+1} = 0\}$ and let $V_{\pm i}$ be open disjoint sets in $\mathbf{R}^{n+1}$ for which $V_{\pm i} \cap \mathbf{R}^n = U_{\pm i}$. Evidently $V = \bigcup\limits_{i=1}^{n} (V_{-i} \cup V_{+i}) \supset \bigcup\limits_{i=1}^{n} (U_{-i} \cup U_{+i}) = \mathbf{R}^n$ since Σ is not essential. Now take a closed ball B in $\mathbf{R}^n$ which contains K. B is a compact set and $B \subset V$ – hence there exists $r > 0$ for which $O_r(B) \subset V$. In particular the cylinder $P = B \times (-r, r)$ lies in V. Let $p \in P$ and form the cone Q with a vertex p and base K: $Q = \bigcup\{[p, x] | x \in K\}$. Clearly $Q \subset P$ because P is a convex set. Furthermore Q is homeomorphic to I^n. A homeomorphism $g : I^n \to Q$ is given for example by the expression

$$g(t) = p + \|t\| \left(f\left(\frac{t}{\|t\|} \right) - p \right)$$

for $t \neq 0$ and $g(0) = p$. Denote now $W_{\pm i} = g^{-1}(Q \cap V_{\pm i})$. Clearly $W_{\pm i} \supset F_{\pm i}$; $W_{-i} \cap W_{+i} = \emptyset$ and if $D_i = I^n \setminus (W_{-i} \cup W_{+i}) = g^{-1}(Q \cap C_i)$ we have $\bigcap\limits_{i=1}^{n} D_i = \emptyset$ which is a contradiction to the essentiality of Σ in I^n.

The family Σ is minimal – this is an easy consequence from the fact that for every proper closed subset L of K there exists an open in K set $U \supset L$ which is homeomorphic to $\mathbf{R}^{n-1}$.

Corollary 3.2. *(The Jordan Curve Theorem) If $K \subset \mathbf{R}^n$ is homeomorphic to S^{n-1}, then $\mathbf{R}^n \setminus K$ contains exactly two connected components.*

Hint of the proof. Consider the $(n-1)$-dimensional sphere S^{n-1} as the boundary of the n-dimensional simplex Δ^n and suppose furthermore that $f : S^{n-1} \to \mathbf{R}^n$ is an embedding. For an arbitrary $\varepsilon > 0$ one can find a simplicial ε-approximation $g_\varepsilon : S^{n-1} \to \mathbf{R}^n$ which is an embedding. Thus for $x \in S^{n-1}$ we have $\|f(x) - g_\varepsilon(x)\| \leq \varepsilon$. According to Lemma 2.4 and Corollary 3.1 $\mathbf{R}^n \setminus g_\varepsilon(S^{n-1})$ consists of two connected open sets – a bounded one B_ε and an unbounded U_ε.

To continue on we divide the points of $\mathbf{R}^n \setminus f(S^{n-1})$ of two sets B and U: $x \in B$ ($x \in U$) if and only if there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the inclusion $x \in B_\varepsilon$ ($x \in U_\varepsilon$) is fulfilled. It is not of serious diffcultness to see that B and U are open and connected and $\mathbf{R}^n \setminus f(S^{n-1}) = B \cup U$.

Corollary 3.3. *([2,3]) Let $K \subset \mathbf{R}^n$ be a compact set. Then $\mathbf{R}^n \setminus K$ is disconnected if and only if there exists an essential map $f : K \to S^{n-1}$.*

Proof. Consider S^{n-1} as a boundary of the n-dimensional cube I^n; $\quad I = [-1, 1]$. Let furthedmore $\Sigma = \{(K_{-i}, K_{+i})\}_{i=1}^n$ be a family of disjoint closed pairs in K. One may take for every i a function $f_i : \mathbf{R}^n \to [-1, +1]$ for which $f_i^{-1}(\pm 1) = K_{\pm i}$ and then put $f(x) = (f_1(x), \ldots, f_n(x))$. It is easy to prove that the restriction of f over K is essential if and only if Σ is an essential family.

Corollary 3.4. *([6]) Every irreducible boundary in $\mathbf{R}^{n+1}$ is a Cantor n-manifold.*

Proof. It is easy to see ([10]) that the fence of every minimal essential n-family in $\mathbf{R}^n$ is a irreducible cyclic compactum in the sense of [6].

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ПОГЛЕД ВЪРХУ ТЕОРЕМАТА НА ЖОРДАН

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В тази работа са описани геометрично абсолютните граници в $\mathbf{R}^n$: подмножеството K на $\mathbf{R}^n$ е абсолютна граница тогава и само тогава, когато е рамка на съществена n-система. Разбира се, резултатът не е нов, но такъв подход не ни е известен. Ползата от него се състои в това, че теоремата на Жордан е тривиално следствие от геометричната характеристика на абсолютните граници.