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FOUR-DIMENSIONAL POINT-WISE HYPERSURFACE OF CONSTANT TYPE

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Following Prof. G. Stanilov we investigate 4-dimensional point-wise hypersurface of constant type in respect to the classical Jacobi operator and to the skew symmetric curvature operator which is defined from him in 1989. Such manifolds are either manifolds of constant sectional curvature or so called parabolical hypersurface without plane points. In the second case one gets also so called hypersurfaces with IP-metric.

In the last 10 years there are many investigations about manifolds of point-wise constant type. There are used usually the following curvature operators:

1. The classical Jacobi operator;
2. The generalized Stanilov-Jacobi operator [1, 2];
3. The skew-symmetric Stanilov curvature operator [1, 3, 4].

But as we know nobody until this time has investigated submanifolds in respect of these three operators. Because of this we are very grateful to Prof. Stanilov for bringing to our attention such problems.

We shall use some facts from [5] for hypersurfaces in the Euclidean space.

The curvature tensor for such a hypersurface M in R^n can be represented by the equations

$$R\left(e_i,e_j\right)e_k=\begin{cases}0 & k\neq i,j\\ -\lambda_i\lambda_je_j & k=i\\ \lambda_i\lambda_je_i & k=j\end{cases}, \tag{1}$$

where λ_i, $i=1,2,\dots,n$ are the eigenvalues of the symmetric operator A (the generalized Weingarten operator in the classical differential geometry of the surfaces) and e_i, $i=1,2,\dots,n$ are the corresponding eigenvalues, which form an orthonormal base. Here we consider dimension $n=4$. In this case the equations (1) can be written more detailed in the following way:

$$
\begin{array}{ll}
R(e_1,e_2,e_1) = -\lambda_1\lambda_2 e_2 & R(e_2,e_3,e_1) = 0 \\
R(e_1,e_2,e_2) = \lambda_1\lambda_2 e_1 & R(e_2,e_3,e_2) = -\lambda_2\lambda_3 e_3 \\
R(e_1,e_2,e_3) = 0 & R(e_2,e_3,e_3) = \lambda_2\lambda_3 e_2 \\
R(e_1,e_2,e_4) = 0 & R(e_2,e_3,e_4) = 0 \\
 & \\
R(e_1,e_3,e_1) = -\lambda_1\lambda_3 e_3 & R(e_2,e_4,e_1) = 0 \\
R(e_1,e_3,e_2) = 0 & R(e_2,e_4,e_2) = -\lambda_2\lambda_4 e_4 \\
R(e_1,e_3,e_3) = -\lambda_1\lambda_3 e_1 & R(e_2,e_4,e_3) = 0 \\
R(e_1,e_3,e_4) = 0 & R(e_2,e_4,e_4) = \lambda_2\lambda_4 e_2 \\
 & \\
R(e_1,e_4,e_1) = -\lambda_1\lambda_4 e_4 & R(e_3,e_4,e_1) = 0 \\
R(e_1,e_4,e_2) = 0 & R(e_3,e_4,e_2) = 0 \\
R(e_1,e_4,e_3) = 0 & R(e_3,e_4,e_3) = -\lambda_3\lambda_4 e_4 \\
R(e_1,e_4,e_4) = \lambda_1\lambda_4 e_1 & R(e_3,e_4,e_4) = \lambda_3\lambda_4 e_3
\end{array} \tag{2}
$$

The sectional curvature of the plane $e_i \wedge e_j$ is given by

$$
K_{ij} = \lambda_i \lambda_j . \tag{3}
$$

I. Investigations of 4-dimensional hypersurfaces in respect to the classical Jacobi curvature operator. Let $M^4 \subset R^5$ be a hypersurface in the five dimensional Euclidean space. The classical Jacobi operator in respect to the unit vector x at the point $p \in M^4$ is defined by the equation

$$
R_x(u) = R(u,x,x), \quad u \in M_p^4.
$$

If the unit vector u is an eigenvector of R_x, then the equation holds good

$$
R(u,x,x) = c(p;x)u.
$$

Here $c(p;u)$ is the corresponding eigenvalue to u which depends of the point p and from the vector u.

If

$$
c(p;x) = c(p) \tag{4}
$$

at any point p of M^4 the hypersurfaces will be called point-wise Osserman hypersurface of constant type.

At first we find a necessary conditions for it.

The operator R_{e_1} in respect to the orthonormal base

e_i, $i = 1,2,3,4$ has the matrix.

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_2 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_4
\end{pmatrix}
$$

It shows that the eigenvalues of R_{e_1} are

$$
0, \quad \lambda_1\lambda_2, \quad \lambda_1\lambda_3, \quad \lambda_1\lambda_4. \tag{5}
$$

In the same way we find: the eigenvalues of the operator R_{e_2} are

$$
0, \quad \lambda_2\lambda_1, \quad \lambda_2\lambda_3, \quad \lambda_2\lambda_4; \tag{6}
$$

of the operator R_{e_3} are

(7) $$0, \quad \lambda_3\lambda_1, \quad \lambda_3\lambda_2, \quad \lambda_3\lambda_4;$$

of the operator R_{e_4} are

(8) $$0, \quad \lambda_4\lambda_1, \quad \lambda_4\lambda_2, \quad \lambda_4\lambda_3.$$

The condition (4) for the point-wise constancy means that the series of the numbers (5)–(8) are identical. Thus we get

Proposition 1. *If the $M^4 \subset R^5$ is point-wise constant Osserman hypersurface, then either the case holds good*
a) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$,
or the case
b) $\lambda_2 = \lambda_3 = \lambda_4 = 0 \neq \lambda_1$
(exactly three of the eigenvalues λ_i are 0, the fourth one can not be 0).

To prove the converse we remark that in the case a) the hypersurface is of constant sectional curvature. In the second case b) all components of the curvature tensor are 0. This case is a generalization of the so-called parabolical surface without plane points in the 3-dimensional Euclidean space. We call such surfaces generalized parabolical hypersurface. Thus we have

Theorem 1. *The hypersurface $M^4 \subset R^5$ is point-wise Osserman hypersurface of constant type iff either it is a hypersurface of constant sectional curvature or it is a generalized parabolical hypersurface, in particular it is also a space of constant (zero) curvature. (eventually Euclidean four-dimensional space).*

II. Investigations of 4-dimensional hypersurfaces in respect to the Stanilov skew-symmetric curvature operator. Now we use the skew-symmetric curvature operator introduced from Prof. Grozio Stanilov in 1989. The definition is the following.

If x, y is any orthonormal pair of vectors in the tangent space, Stanilov's curvature operator is defined by

$$k_{x,y}(u) = R(x,y,u), \quad u \in Mp$$

$k_{x,y}$ does not depend of the orthonormal base x, y of the plane $E^2 = e_i \wedge e_j$. Because of this it is defined

$$k_{E^2} = k_{x,y}\,.$$

This operator is skew-symmetric one. Because of this fact it has not real eigenvalues. But its square has real eigenvalues. More detailed, if

$$k_{E^2}(u) = c(p;E^2)\sqrt{-1}\ u, \quad u \in Mp,$$

then

$$k^2_{E^2}(u) = -c(p,E^2)^2\ u, \quad u \in Mp.$$

A hypersurface M^4 is called point-wise Stanilov of constant type iff

$$c\left(p;E^2\right) = c\left(p\right)$$

for any 2-dimensional plane E^2 in Mp for any $p \in M$.

Using [6] we have

$$k_{e_i,e_j}(u) = -K^2_{ij}\ u, \quad u \in e_i \wedge e_j.$$

For any $e_i \wedge e_j$ we have

$$K_{ij}^2 = (\lambda_i \lambda_j)^2$$

The condition of the constancy of the eigenvalues of this operator means that the following equations hold good:

$$\lambda_1^2\lambda_2^2 = \lambda_1^2\lambda_3^2 = \lambda_1^2\lambda_4^2 = \lambda_2^2\lambda_3^2 = \lambda_2^2\lambda_4^2 = \lambda_3^2\lambda_4^2.$$

Essentially we have two different cases:

1. At least two of λ_i^2, λ_j^2 are different. If $\lambda_s^2 \neq \lambda_k^2$ it follows that all $\lambda_i = 0$ except eventually one of them is not 0. For example $\lambda_1 = \lambda_2 = \lambda_3 = 0 \neq \quad \lambda_4$.

2. If all λ_i^2 are equal, then we have essentially the following three algebraic possibilities for the signs of the numbers, which are modulo equal $(|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4|)$:

	λ_1	λ_2	λ_3	λ_4
2.1	+	+	+	+
2.2	+	+	+	−
2.3	+	+	−	−

Then:

2.1) $K_{12} = K_{13} = K_{14} = K_{23} = K_{24} = K_{34}$
and the hypersurface is of constant sectional curvature.

2.2) $K_{12} = K_{13} = K_{23} = -K_{14} = -K_{24} = -K_{34}$.

In this case the hypersurface has IP-metric [7] (case (v) from Proposition 2.3).

2.3) $K_{12} = K_{34} = -K_{13} = -K_{14} = -K_{23} = -K_{24}$.

This case is impossible (the case (iv) from the same Proposition 2.3 [7]).

Thus we can formulate the following

Theorem 2. *A four dimensional hypersurface M^4 in the 5-dimensional Euclidean space R^5 is point-wise Stanilov of constant type if it is one of the following three types:*

I. The hypersurface M^4 is of constant sectional curvature;

II. The hypersurface M^4 is a generalized parabolical hypersurface (in particular it is also a space of constant (zero) curvature).

III. The hypersurface M^4 is a warped product: $B^1 \times_f N^3$;

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4-МЕРНИ ТОЧКОВО ПОСТОЯННИ ХИПЕРПОВЪРХНИНИ ОТ КОНСТАНТЕН ТИП

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Следвайки проф. Станилов разглеждаме 4-мерни точково постоянни хиперповърхнини от константен тип на R^5 по отношение на класическия оператор на Якоби и на антисиметричния кривинен оператор дефиниран от него през 1989. Такива хиперповърхнини са с постоянна секционна кривина. Сред тях са и така наречените параболични повърхнини без равнинни точки (които също са с постояна (нулева) секционна кривина). При антисиметричния оператор се получават и хиперповърхнини с IP-метрика (с метрика на Иванов-Петрова).