МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2002
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2002
Proceedings of Thirty First Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 3–6, 2002

CHARACTERIZATION OF A FOUR-DIMENSIONAL EINSTEIN-RIEMANNIAN MANIFOLDS BY DEGENERATED JACOBI OPERATOR AND BASISES OF SINGER-THORPE

Veselin T. Videv, Yulian T. Tzankov, Maria V. Stoeva

In the present short note we charactreize these classes of a four-dimensional Einstein-Riemannian manifolds, for which the determinant of Jacobi operator R_X is equal to zero at any point of the manifold, and for any tangent vector X, which belongs to one of the coordinate planes spanned from a basis of Singer-Thorpe.

Let (M, g) be an n-dimensional Riemannian manifolds with metric tensor g, let M_p be the tangent space at a point $p \in M$, and let S_pM be the set of all unit tangent vectors in M_p. If ∇ is the *Levi-Civita* connection of M, then R is the curvature tensor defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for any tangent vectors $X, Y \in M_p$, where $p \in M$. Let $\rho(x, y)$ be the *Ricci bilinear function* defined by $\rho(x, y) = \text{trace } (u \to R(u, x, y))$, for any $x, y \in M_p$, $p \in M$, let ρ be the corresponding *Ricci operator* defined by $g(\rho(x), y) = \rho(x, y)$, and let τ be the trace of ρ which is called scalar curvature on M[1]. It is well-known the following

Theorem 1[1]. *Let (M, g) be an n-dimensional $(n \geq 3)$ Riemannian smooth manifold such that the Ricci curvature $\rho(X, X)$ does not depend on any tangent vector $X \in S_pM$, at any point $p \in M$. Then*

$$\rho(X, X) = \lambda, \quad \textit{where } \lambda \textit{ is a constant on } M. \tag{1}$$

Definition 1 [6]. *Any Riemannian manifold (M, g) with property (1), is called Einstien-Riemannian manifold, such that*

$$\rho(X, Y) = \lambda g(X, Y), \ \textit{for any tangent vectors } X, Y \in M_p, \ \textit{at any point } p \in M. \tag{2}$$

It is well-known that the corresponding curvature tensor of type (0,4) has the following algebraic properties [1]:

$$\begin{gathered} R(x, y, z, u) = -R(y, x, z, u), \\ R(x, y, z, u) = -R(x, y, u, z), \\ R(x, y, z, u) + R(y, z, x, u) + R(z, x, y, u) = 0 - \text{ first Bianchi identity.} \end{gathered} \tag{3}$$

Also R satisfies the second differentially Bianchi identity [1]:

$$\sigma_{xyz}(\nabla_x R)(y, z, u) = 0, \tag{4}$$

where σ is a cyclic sum over x, y, z. Let $\wedge^2(M_p)$ be a 2-vector space over M_p with standard metric [2]: $\hat{g}(v_1 \wedge v_2, w_1 \wedge w_2) = \det(g(v_i, w_j))$ $(v_1,\ v_2,\ w_1,\ w_2 \in M_p)$ and with curvature operator $\Re$ defined by [2]: $\hat{g}(\Re(x \wedge y), z \wedge v) = g(R(x, y, z), v) = R(x, y, z, v)$, $x, y, z, v \in M_p$. If $e_1, e_2, \ldots, e_n$ is an orthonormal basis in M_p, then 2-vectors $e_1 \wedge e_2, e_1 \wedge e_3, \ldots, e_1 \wedge e_n, \ldots, e_{n-1} \wedge e_n$ formed a reducible orthonormal basis in $\wedge^2(M_p)$ [2].

In the sequel let dim $M = 4$. Let $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6$ are eigen 2-vectors of $\Re$ with the corresponding eigenvalues $a_1, a_2, a_3, a_4, a_5, a_6$. Then the matrix of $\Re$ with respect to the reducible orthonormal 2-vector basis $P_1, P_2, P_3, P_1^\perp, P_2^\perp, P_3^\perp$ in $\wedge^2(M_p)$, defined by formulas

$$P_i = \frac{\xi_i + \xi_{i+3}}{\sqrt{2}}, \quad P_i^\perp = \frac{\xi_i - \xi_{i+3}}{\sqrt{2}}, \quad i = 1, 2, 3, \tag{5}$$

has the form

$$(P) = \begin{pmatrix} \lambda_1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & \mu_3 \\ \mu_1 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & \lambda_3 \end{pmatrix}, \tag{6}$$

where

$$\lambda_i = \frac{a_i + b_i}{2}, \quad \mu_i = \frac{a_i - b_i}{2}, \quad (i = 1, 2, 3)\ [6]. \tag{7}$$

Because of $P_1, P_2, P_3, P_1^\perp, P_2^\perp, P_3^\perp$ are a reducuble 2-vectors, then there exist an orthonorlmal basis η_1, η_2, η_3, η_4 of tangent vectors in M_p, such that [2]:

$$P_1 = \eta_1 \wedge \eta_2, P_2 = \eta_1 \wedge \eta_3, P_3 = \eta_1 \wedge \eta_4, P_1^\perp = \eta_2 \wedge \eta_3, P_2^\perp = \eta_2 \wedge \eta_4, P_3^\perp = \eta_3 \wedge \eta_4. \tag{8}$$

All these results are sumarized in

Theorem 2 [6]. *Let (M, g) be a 4-dimensional Einstein-Riemannian manifold. Then at any point $p \in M$ there exist an orthonormal basis $\eta_1, \eta_2, \eta_3, \eta_4$ in the tangent space M_p defined by (8), such that the matrix of the curvature operator $\Re$ with respect to this basis has the form (6).*

This basis was called Singer-Thorpe basis [5]. We can connect with any point p of an Einstein-Riemannian manifold (M, g) a Singer-Thorpe basis, so that all esentaily components of the curvature tensor R with respect to this basis can be expressed by the formulas [5]:

$$\begin{aligned} &R_{1221} = R_{3443} = \lambda_1, R_{1331} = R_{2442} = \lambda_2, R_{1441} = R_{2332} = \lambda_3, \\ &R_{1234} = \mu_1, R_{1342} = \mu_2, R_{1423} = \mu_3. \end{aligned} \tag{9}$$

Definition 2. *We denote by Π_p the surface uniting all coordinate planes $\eta_i \wedge \eta_j$ of a Singer-Thorpe basis $\eta_1, \eta_2, \eta_3, \eta_4 \in M_p$, at a point p of an Einstein-Riemannian manifold (M, g).*

These planes are an extremal values of the function of the sectional curvature K on M. In the sequel using these surfaces, we will obtain a generalization of the pointwise Osserman condition of the Jacobi operator R_X, which is defined as a symmetric endomorphism of the tangent space M_p, for a tangent vector $X \in S_pM$, at a point p of a Riemannian manifold (M,g), by $R_X(u) = R(u,X,X)$ [4]. It is easy to prove that trace $R_X = \rho(X)$, where ρ is the Ricci tensor of M. When trace R_X is a pointwise function on (M,g), then according to Theorem 1 we obtain that (M,g) is an Einstein-Riemannian manifold. If (M,g) is a four-dimensional Riemannian manifold, then the characteristically equation of any Jacobi operator R_X can be written in the form $(c^3 - J_1c^2 + J_2c - J_3) = 0$, where $J_1 = \mathrm{trace} R_X$ and $J_3 = \det R_X$. Using the characteristically matrix of R_X with respect to an arbitrary orthonormal basis e_1, e_2, e_3, e_4 in $M_p, p \in M$, we obtain:

$$(10)\quad \begin{aligned}
&\det R_{ae1+be2} = J_3(p; ae_1+be_2) = a^4(-2R_{2113}R_{3114}R_{2114}+K_{12}K_{13}K_{14}-K_{12}R^2_{3114}\\
&-K_{13}R^2_{2114}-K_{14}R^2_{2113}) + 2a^3b(R_{2114}R_{2113}(R_{3124}+R_{3214}) - R_{2113}R_{3114}R_{1224}\\
&-R_{1223}R_{3114}R_{2114} + K_{12}K_{14}R_{1332}+K_{12}K_{13}R_{1442}-K_{12}(R_{3124}+R_{3214})R_{3114}\\
&-R_{1332}R^2_{2114} + K_{13}R_{2114}R_{1224} + K_{14}R_{2113}R_{1223}-R_{1442}R^2_{2113})\\
&+a^2b^2(K_{12}K_{13}K_{24} + 4K_{12}R_{1442}-2R_{1224}R_{2113}(R_{3124}+R_{3214}) + K_{12}K_{14}K_{24}\\
&+2R_{1223}R_{3224}R_{2114} - 2R_{1223}R_{3114}R_{1224} - 2(R_{3124}+R_{3214})R_{1223}R_{2114} - K_{23}R^2_{2114}\\
&+4R_{1332}R_{2114}R_{1224} - K_{13}R^2_{1224} - K_{12}(R_{3124}+R_{3214})^2 - 2K_{12}R_{3114}R_{3224}\\
&-K_{24}R^2_{2113} + 4R_{1442}R_{2113}R_{1223}-K_{14}R^2_{1223}) + 2ab^3(K_{12}K_{23}R_{1442}+K_{12}K_{24}R_{1332}\\
&-R_{2113}R_{3224}R_{1224} - R_{1224}R_{1223}(R_{3124}+R_{3214}) - R_{2114}R_{1223}(R_{3124}+R_{3214})\\
&+K_{23}R_{2114}R_{1224} - R^2_{1224}R_{1332} - K_{12}R_{3224}(R_{3124}+R_{3214}) + K_{24}R_{1223}R_{2113}\\
&-R^2_{1223}R_{1442}) + b^4(K_{12}K_{23}K_{24}-2R_{1223}R_{3224}R_{1224} - K_{12}R^2_{3224} - K_{23}R^2_{1224}\\
&-K_{24}R^2_{1223}),
\end{aligned}$$

for any real numbers a, b, where $a^2+b^2 = 1$. Analogously, after a cyclic permutation of the indices in (10), we can obtain the corresponding expressions of $\det R_{ae_i+be_j}$, for any $i,j = 1,2,3,4$. Using this expresions, the equations $\det R_{ae_i+be_j} = 0$, and (9), we prove the following

Lemma 1. *Let (M,g) be a 4-dimensional Einstein-Riemannian manifold. If* det $R_X = 0$, *for any tangent vector $X \in \Pi_p$, at any point $p \in M$, then for the invariants $\lambda_1,\ \lambda_2,\ \lambda_3, \mu_1,\ \mu_2,\ \mu_3$ of the Singer-Thorpe basis in the tangent space M_p, we have one of the following formulas:*

$$(11)\qquad 4\lambda_1 = \tau, \text{ and } \lambda_s = \mu_s = 0, \text{ for } s = 2,3, \text{ and } \mu_1 = 2\mu_2 = 2\mu_3;$$

$$(12)\qquad \lambda_k = \mu_k = 0 \text{ or } \lambda_i - \lambda_j = \pm(\mu_i - \mu_j), \text{ for any } i \neq j;\ i,j,k = 1,2,3,4.$$

First we will consider case (11). Using the components of the metric tensor g, with respect to an orthonormal basis $e_1, e_2, e_3, e_4 \in M_p$, we obtain the formula

$$(13)\quad \begin{aligned}
R(x,y,z,u) \ &= \lambda_1.\Big((g_{x1}.g_{y4} - g_{x4}.g_{y1}).(g_{z4}.g_{u1} - g_{z1}.g_{u4})\\
&+(g_{x2}.g_{y3} - g_{x3}.g_{y2}).(g_{z3}.g_{u2} - g_{z2}.g_{u3})\Big)\\
&+\mu_3.\Big((g_{x1}.g_{y2} - g_{x2}.g_{y1}).(g_{z3}.g_{u4} - g_{z4}.g_{u3})\\
&+(g_{x3}.g_{y4} - g_{x4}.g_{y3}).(g_{z1}.g_{u2} - g_{z2}.g_{u1})\\
&+(g_{x1}.g_{y3} - g_{x3}.g_{y1}).(g_{z4}.g_{u2} - g_{z2}.g_{u4})\\
&+(g_{x4}.g_{y2} - g_{x2}.g_{y4}).(g_{z1}.g_{u3} - g_{z3}.g_{u1})\\
&-2((g_{x1}.g_{y4}-g_{x4}.g_{y1}).(g_{z2}.g_{u3} - g_{z3}.g_{u2})\\
&+(g_{x2}.g_{y3}-g_{x3}.g_{y2}).(g_1.g_{u4}-g_{z4}.g_{u1}))\Big),
\end{aligned}$$

where $x, y, z, u \in M_p$, at a point $p \in M$. Putting $z = e_1, u = e_2$ in (13) we get

$$R(x, y, e_1, e_2) = \mu_3.(g_{x3}.g_{y4} - g_{x4}.g_{y3}). \tag{14}$$

Further we suppose that the tangent vectors $x, y, z, u, e_1, e_2, e_3, e_4$ can be continued as a smooth tangent vector fields in a neighborhoud $U_p \subseteq M_p$, so that at any point $q \in U_p, e_1, e_2, e_3, e_4$ form a Singer-Thorpe basis in M_q. Using (14), and some properties of ∇ [1], we have in index form the following relation:

$$\begin{aligned}\nabla_z R_{xy12} &= \lambda_1.\nabla_z(g_{x4}.g_{y3} - g_{x3}.g_{y4}) + (g_{x3}g_{y4} - g_{x4}.g_{y3}).\nabla_x\lambda_1 \\ &+(g_{z3}.g_{x4} - g_{z4}.g_{x3}).\nabla_z\mu_3 + (g_{y3}.g_{z4} - g_{y4}.g_{z3}).\nabla_y\mu_3 \\ &+\mu_3\Big(g(\nabla_z y, e_3).g_{x4} + g(\nabla_z y, e_4).g_{x3}) + (g_{x3}.g_{y4} - g_{x4}.g_{y3}) \\ &-g(\nabla_y z, e_3).g_{x4} - g(\nabla_y z, e_4).g_{x3} - g(\nabla_x y, e_3).g_{z4} + g(\nabla_x y, e_4).g_{z3} \\ &-g(\nabla_x z, e_3).g_{y4} + g(\nabla_x z, e_4).g_{y3} - g_{y3}.g_{z4} + g_{y4}.g_{z3} - g(\nabla_y x, e_3).g_{z4} \\ &+g(\nabla_y x, e_4).g_{z3} - g_{z3}.g_{x4} + g_{z4}.g_{x3} + \nabla y(g_{z4}.g_{x3} - g_{z3}.g_{x4})\Big) \\ &-g_{y3}\Big(g(\nabla_z x, e_3).g_{y4} - g(\nabla_z x, e_4)\Big)g_{y3} + \nabla_x(g_{y4}.g_{z3} - g_{y4}.g_{z4}).\end{aligned} \tag{15}$$

Since from (4) we have the equation $\sigma_{XYZ}(\nabla_X R)(y, e_1, e_2) = 0$, where σ is a cyclic sum over x, y, z, then using (11), (15) and putting $x = e_3, y = e_4, z = e_1, x = e_3, y = e_4, z = e_1$ we obtain

$$-\nabla_{e_1+e_2}(\mu_3) - (g(\nabla_{e_1}e_4, e_4) - g(\nabla_{e_1}e_3, e_3) + g(\nabla_{e_3}e_1, e_3) + g(\nabla_{e_4}e_1, e_4) - 1) = 0. \tag{16}$$

Here changing e_2 by $-e_2$ we get a new equation:

$$-\nabla_{e_1-e_2}(\mu_3) - (g(\nabla_{e_1}e_4, e_4) - g(\nabla_{e_1}e_3, e_3) + g(\nabla_{e_3}e_1, e_3) + g(\nabla_{e_4}e_1, e_4) - 1) = 0. \tag{17}$$

From the equations (16) and (17) we have $\nabla e_1(\mu_3) = 0$, and analogously we can prove that $\nabla e_i(\mu_3) = 0$, for any $i = 1, 2, 3, 4$. From here and from some linear properties of the Levi-Civitta connection ∇ [1], it follows that $X(\mu_3) = 0$, for any tangent vector $X \in U_p$. That means that μ_3 is a constant on U_p, and now from (14) it follows directly that $\mu_3 = 0$. Hence if (11) are satisfied, we have at all

$$4\lambda_1 = \tau = \text{const}, \quad \lambda_2 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 0, \tag{18}$$

for the invariants of any Singer-Thorpe basis in M_q, at any point $q \in U_p$. Further using (18) we'll prove that (M, g) is a reducible space on U_p. On definition *an n-dimensional Riemanniann manifold (M, g) is a reducible manifold on U_p, if there exist a coordinate system x_1, x_2, x_3, x_4 in U_p, such that the metric tensor g in U_p can be represented in the form $ds^2 = \Sigma_{i=1,r}\varphi_k$, where φ_k are quadratic forms which depend on r variables, where* $1 < r < n$ [3]. According to this definition we have that if dim $M = 4$, then there are possible two types of metrics in M_p:

$$ds^2 = \varphi_1(x_1) + \varphi_2(x_1, x_2, x_3), \tag{19}$$

$$ds^2 = \varphi_1(x_1, x_2)\varphi_2(x_3, x_4). \tag{20}$$

If we have (18) for the invariants of Singer-Thorpe basis $e_1, e_2, e_3, e_4 \in M_q$, at any point $q \in U_p, p \in M$, then we have and the following system of differential equations: $R_{ijkl,s} = 0$. This system has solutions if and only if there exist a constant symmetric tensor field T of type $(0, 2)$, defined on U_p, which satisfies a linear system $R^k_{ijs}.T_{kt} + R^k_{tjs}.T_{ik} = 0$ [3]. Putting $T_{it,[js]} := R^k_{ijs}.T_{kt} + R^k_{tjs}.T_{ik}$, we obtain such a tensor field T, which has also the properties:

$$T_{ij,k} = 0, \tag{21}$$

$$T_{[i,j]} = 0, \tag{22}$$

$$T_i^k . T_{kj} = T_{ij}. \tag{23}$$

From these properties it follows that the entries of the matrix of T, with respect to the Singer-Thorpe basis $e_1, e_2, e_3, e_4 \in M_q$, at any point $q \in U_p$, $p \in M$, satisfy the equations $T_{ij} = 0$ $(i \neq j)$, and $T_{11}^2 + T_{11} = T_{33}^2 + T_{33} = 0$. From here it is easy to check that rank $(T) = 2$, which means that for the metric form in M_q, at any point $q \in U_p$, we have the case (20). More exactly [3]:

$$\varphi_1(x_1, x_2) = dx_1^2 + \cos^2\left(\sqrt{\lambda} x_1\right) dx_2^2 + dx_3^2 + \cos^2\left(\sqrt{\lambda} x_3\right) dx_4^2; \lambda > 0; \tag{24}$$

$$\varphi_2(x_3, x_4) = dx_1^2 + \mathrm{ch}^2\left(\sqrt{-\lambda} x_1\right) dx_2^2 + dx_3^2 + \mathrm{ch}^2\left(\sqrt{-\lambda} x_3\right) dx_4^2; \lambda < 0. \tag{25}$$

Let for the invariants of Singer-Thorpe basis formulas (12) are satisfy. Then according to the well-known result of Sekigava-Vanhecke[5] we have that (M, g) must be a poinwise Osserman manifold, or (M, g) is flat, at any point of M. For this class of manifolds we have

Definition 3 [4]**.** *A Riemannian manifold (M, g) is called a pointwise (globally) Osserman manifold if the eigenvalues of Jacobi operator R_X are pointwise (globally) constants functions for any $X \in S_pM$, at any point $p \in M$.*

The example for a pointwise Osserman manifold which is not globally Osserman manifold was given in [4]. It is a Riemannian manifold (M, g) for which there exist a Clifford module structure $J_1, \ldots, J_k$ in the tangent space M_p, at any point $p \in M$. From this example, by $k = 1$, we obtain a subexample when any Jacoby operator R_X has two zero and two non-zero eigenvalues.

Now all our result above we can summarize in the main result

Theorem 3. *Let (M, g) be a four-dimensional Einstein-Riemannian manifold. Then det $R_X = 0$, for any tangent vector $X \in \Pi_p$, at any point $p \in M$, if and only if or (M, g) is a pointwise Osserman manifold, or (M, g) is a 4-dimensional reducible space with the metrics defined by* (20), (24) *and* (25)*, on some neighborhoud U_p, at any point $p \in M$.*

In the end of the paper we remark that the converse result of Theorem 3 is also true, but we do not prove it because it is trivial.

REFERENCES

[1] Гр. Станилов. Диференциална геометрия. София, Наука и изкуство, 1988.

[2] А. Мальцев. Основы линейной алгебры. Наука, Москва, 1970.

[3] П.Широков. Симетрические конформно-евклидовы пространства. *Изв. Казанск. физ.-мат. об-ва*, сер. 3 (1938), 11.

[4] P. Gilkey, A. Swann, L. Vanheke. Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator. *Quart. J. Math. Oxford* (1995), 299–320.

[5] K. Sekigawa, L. Vanhecke. Volume preserving geodesic symmetries in four-dimensional 2-stein spaces. *Kodai. Math J.*, **9** (1986), 215–223.

[6] I. Singer, J. Thorpe. The curvature of 4-dimensional Einstein spaces. Global Analisys. Papers in Honour of K. Kodaira. University of Tokio press, Princeton University Press, 1969, 354-365.

Julian Tzankov Tzankov
Faculty of Mathematics and Informatics
Department of Geometry
Sofia University "St. Kl. Ohridski"
5, James Baucher Blvd.
1164 Sofia, Bulgaria,
e-mail: ucankov@fmi.uni-sofia.bg

Veselin Totev Videv
Department of Mathematics and Physics
Trakia University
Students town
6000 Stara Zagora
Bulgaria
e-mail: videv@uni-sz.bg

Maria Valkova Stoeva
Technical College, Trakia University
38, Graph Ignatiev Str.
Yambol, Bulgaria
e-mail: yacol@descom.com

ХАРАКТЕРИЗИРАНЕ НА ЧЕТИРИМЕРНИ АЙНЩАЙНОВИ РИМАНОВИ МНОГООБРАЗИЯ ЧРЕЗ ИЗРОДЕН ОПЕРАТОР НА ЯКОБИ И БАЗИСИ НА СИНГЕР-ТОРП

Веселин Т. Видев, Юлиан Ц. Цанков, Мария В. Стоева

В представената статия характеризираме тези класове четиримерни Айнщайнови-Риманови многообразия за които детерминантата на оператора на Якоби R_X е равна на нула във всяка точка от многообразието и за всеки допирателен вектор X принадлежащ на координатна равнина на базис на Сингер-Торп.