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MATHEMATICS IS BORING? WHAT CAN WE DO? TESSELLATIONS AS ONE OF THE POSSIBLE SOLUTIONS*

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This article deals with introducing the notion of tessellating the plane in the mathematics classes as one of the possible ways of motivating the students to learn about some important mathematical phenomena such as the properties of the regular polygons, the plane transformations, etc. The tessellation of the plane is given both as a way to introduce these mathematical topics in the traditional classroom and in the context of applying new information technologies.

Mathematics has always been one of the most controversial sciences. While some people treated it as the "queen of the sciences", for most of the rest it was non-understandable, boring and unnecessary. The present situation is not changed at all. On the contrary, mathematics is considered to be one of the most difficult subjects in school whose real meaning is hard to find, but which, unfortunately, students have to pass. We, the mathematics educators, have to face that problem as fast as we can and try to find strategies to make people appreciate mathematics in a way similar to what professional mathematicians do. We have to DO SOMETHING and show to people that mathematics is not a science for its own sake, i.e. it is not isolated and non-applicable, but, on the contrary, it is everywhere around us and everyone applies it, consciously or not. That's why, when we introduce a new notion, it would be better to find some interesting examples from other fields where the same notion has been applied. Thus the introduction of new notions may become more motivated and the students will be aware of their applications. Of course, for that purpose, we need an interdisciplinary collaboration, but this is necessary if we want the public opinion to accept the mathematics as a basic instrument in everyday life. The best start for such a strategy is the classroom and it wouldn't be in a conflict with the rigorous and precise mathematics, but an introduction to it.

For example, it would be much easier first to derive the formula for the sum of the interior angels of the regular polygons and only then to generalize it for the case of an arbitrary polygon. For that purpose we can use the notion of the *tessellation of the plane* i.e. the tiles that cover the plane without gaps or overlapping. After introducing the notion of *regular polygons*, the teacher could show to the class pictures similar to fig. 1 and ask the questions as follows:

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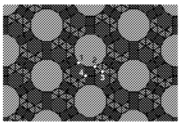

Fig. 1

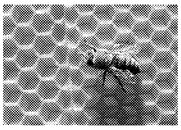

Fig. 2

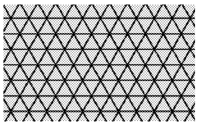

Fig. 3

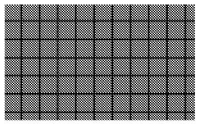

Fig. 4

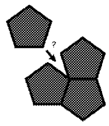

Fig. 5

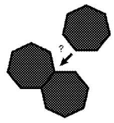

Fig. 6

Which regular polygons do you see on the picture? Are they overlapping? Are there gaps between them?

Thus the teacher could present the notion of *tessellation* and then ask the students for some examples of tessellation from everyday life.

We could find some examples of tessellation of the plane everywhere around us: in the nature, in the streets, on the clothes, on the carpets, on the fancy, on the columns, walls and floors of the buildings from the ancient times till today. A typical example of tessellation in the nature are the honeycombs of the bees whose cells are all equal, adjacent and hexagonal in form (fig. 2). Students could easily realize that there are exactly three regular hexagons in each vertex and it wouldn't be hard to find the sum of hexagon's interior angels. Why the bees do not use some other regular polygons for their cells is another interesting topic to be considered.

Let's start with the equilateral triangle. Its angles are less than the angles of the regular hexagon, so students should realize that around one point in the plane, they can arrange exactly six equilateral triangles without overlapping. If the bees make triangular cells of the same size as the hexagons (fig. 3), they should use much more material.

As the great Pappus of Alexandria had noted:

Bees ... by virtue of a certain geometrical forethought ... know that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material.

What's "wrong" with the square? There are four squares around one point in the plane (fig. 4) but if students try to cover the same area with regular hexagons and compare the material to be used in both cases, they will figure out that the square will be also more "expensive" for the bee's purpose.

What will happen with the regular pentagon? We can arrange three regular pentagons around one point in the plane, without overlapping (fig. 5), but in that case there is one triangle left. So, the regular pentagon is a figure, which does not tessellate the plane. The bees seem to know that ...

If the students try to tessellate the plane with a regular polygon of more than six sides (fig. 6), they will easily see that it is not possible to use more than two figures because their angles are bigger than the angles of the regular hexagon. In that case the tessellation should contain additional type of polygons and thus will be more complicated. The bees should use again more material, which would be very bad for their "budget".

So, the students could easily conclude that the only regular polygons, which tessellate the plane, are: triangle, square and hexagon. After this, the students would hopefully take an active part in deriving the formula for the sum of the interior angles of the regular polygons. Now, the teacher could ask questions of the following kind:

Can we tessellate the plane with regular polygons only? Can you find such examples in the school? How can we determine the sum of the interior angles in the arbitrary polygons?

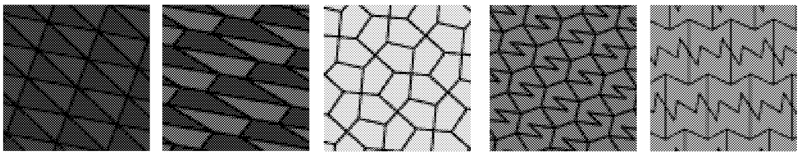

Fig. 7

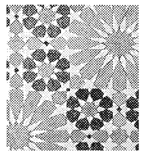

Fig. 8

The teacher could show, as it is shown in fig. 7, tessellation of different arbitrary polygons and ask the students to "discover" the formula for the sum of the interior angles of the convex polygons.

But it is interesting to note that tessellation of the plane has been used in art, too. Furthermore, it would be very motivating for students with deep interest in art to learn about some painters who have applied the idea of tessellation in their works. One of the most remarkable is the Dutch artist M.C. Escher who was amazed by the mosaics of the Alhambra palace in Spain (fig. 8) and started to investigate the tessellation of the plane. One of his best friends were Polya and Coxeter and he learned a lot of them thus giving a very precise mathematical explanation of his paintings. Provoked by the fact that Arabian artists, because of their religion, used only geometric figures as tiling, he started to experiment with stylized models of people, animals or other objects using symmetry, translation, rotation and optical reflection. Thus, we could use the tessellation in the Escher's paintings to illustrate better the notions being introduced. How does he succeed to make the tessellation with stylized models of the living beings? Basically, he uses regular polygons and by changing their form appropriately he gets the wanted shape which can be seen in fig. 9 (it is made with Logo procedure).

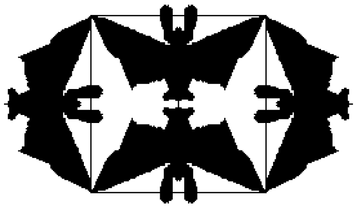

Fig. 9

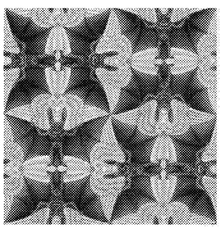

Fig. 10

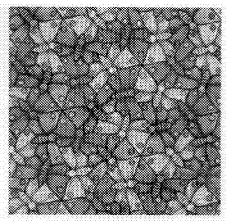

Fig. 11

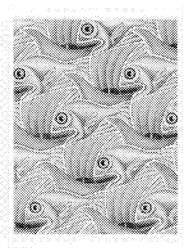

Fig. 12

To make the figure which tessellates the plane in this case we start with the square, we cut out a specific shape from the one side of the square and then add it in the same position at the other side of the square i.e. we perform a translation of the shape. In this case we get a figure of two bets and two angels. Fig. 10 is the Escher's painting called *Angels & Devils* in which he use fig. 9 as a tile.

Now, the following question arises: *Can we become a new Escher?* i.e. if there is some figure in the plane, which mathematical instruments we should use so as to make figure the basic element of the tessellation. This problem is known as *Escherizaton problem.*

Before deriving the notion of rotation, the teacher could show the painting *Butterflies* (fig. 11) where Escher tessellate the plane by rotating butterflies three times. The teacher could ask the questions of the kind:

Which polygon has been transformed?

If we have the same picture on the folio and if we overlap both picture to be coincident how can the butterfly with one color reach the other two positions?

What angle the butterfly make in that way?

The painting *Angels & Devils* (fig. 10) is an excellent example of symmetry. Now, the teacher could ask questions like:

Which polygon has been transformed?

Which pattern can you notice? According to which line?

Are there any other transformations used in this paintings?

Before introducing the notion of translation, the teacher could show paintings similar to *Fish & Boats* (fig. 12) where Escher tessellates the plane by translations. The students could again consider questions like: *which polygon has been transformed, which pattern can they notice, are the figures equally distant, with the same heading and direction and are there any other transformations used?* Actually, almost all Escher's paintings are made by using a composition of these transformations.

Such an introduction of new notions makes them more appealing to the students and has an additional educational effect bringing knowledge from other domains.

Things become even more exciting if we use appropriately computers in the teaching of mathematics. The computer language Logo is a powerful educational environment, which enhances the creativity of the students and helps to reshape the traditional way of acquiring the mathematical notions. In such an exploratory computer environment the students can carry out experiments navigated by the teacher who is also in the role of investigator. They usually "discover" a lot of interesting results, some of which might occur due to their mistakes. If the teacher has the possibility to use Logo he could ask the students to make a program for tessellation of the plain with a regular polygon first, and then to create their own figure for tessellation. What is the use of that? Let's consider fig. 13, which is fig. 1, made by a Logo procedure. To make a program for drawing the figure the students have:

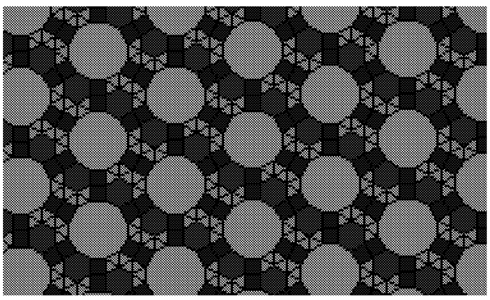

Fig. 13

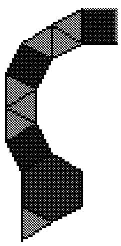

Fig. 14

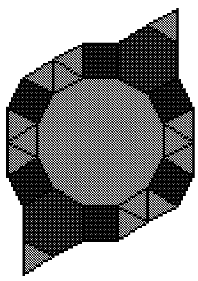

Fig. 15

1. To determine the basic design element of the tile pattern. Thus they develop their imagination, abstract thinking and their ability to analyze.

2. Define a Logo procedure for constructing the basic design element of the tile pattern (fig. 14). In this case the students should construct the regular polygons by using the translations (with the **forward** command) and rotation (with the **right** command). In addition they should determine the exterior angle of the regular polygons because the Logo turtle rotates with respect to its own heading.

3. To construct the tile pattern (fig. 15) switching between translations and rotation.

4. To make the tessellation by using translations.

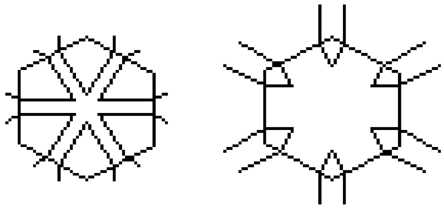

Fig. 16

If we consider a Logo procedure of fig. 9 we can use it for explaining symmetry according to the turtle's heading.

Some other tile patterns as, e.g. those in fig. 16 ([1]), need the knowledge of trigonometry.

In such a way we can provoke the curiosity of students who are not very interested in mathematics because there is no "hard" theory behind tessellation, but the results obtained are very interesting and could be a result of an individual or a group work. Further on, the teacher could develop the following topics:

- *Which triangles and quadrilaterals tessellate the plane?* [7]
- *How many different types of pentagons and hexagons tessellate the plane?* [2]
- *How many different ways are there to tessellate the plane with two or more regular polygons, so that the arrangement of polygons around each vertex is the same?* [2]
- *How can tessellation be used as music generators?* [3]
- *Could the chess table be with other polygons than squares?* [4]
- *How can we tessellate the space?* [2]
- *There are a lot of interesting problems of combinatorial nature* [5]

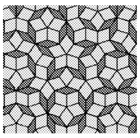

Fig. 17

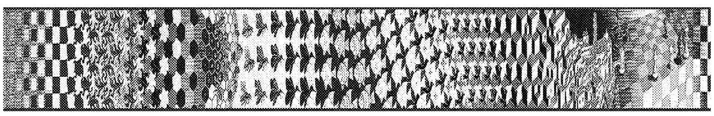

Fig. 18

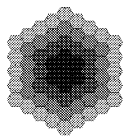

Fig. 19

While working with the talented students the teacher could discuss the subject of *nonperiodic tessellation* and *parquet deformations*. A nonperiodic tessellation is a tessellation in which there is no regular repetition of the tiling. The well-known mathematician Roger Penrose succeeded to find a set of two tiles only that could tessellate the plane nonperiodically (fig. 17). A meta-composer of parquet deformations is William Huff, *a professor of architectural design. He has elicited a hundreds of them,* but he has never executed one [4]. They are different from Escher's transformation (fig. 18 called *Metamorphoses II*) because there is a single type of a tile undergoing consecutive transformations. *The deformations are not arbitrary but must satisfy two basic requirements:*

(*i*) *There shall be change only in one dimension, so that one can see a temporal progression in which one tessellation gradually becomes another;*

(*ii*) *At each stage, the pattern must constitute a regular tessellation of the plane (i.e., there must be a unit cell that could combine with itself so as to cover an infinite plane exactly)* [4].

Also, there are some open problems of interest, e.g. the so-called *Heesch's problem* that reads: *How many times can a tile be surrounded by congruent copies of itself? That is, how many* ***layers*** *made of copies of the tile can you place around the tile. The layers are called* ***coronas****, and the maximum number of coronas that can surround the tile is called the* ***Heesch number*** *of the tile* [8]. The fig. 19 is with Heesch number four. The largest known finite Heesch number is five. So, there's still no answer of the question*: Is there a maximum Heesch number*? [8].

Heesch's problem has connections to a couple of still unsolved problems – the domino problem and the "Einstein" problem (ein = one, stein = tile). *The domino problem asks if there exists an algorithm that , when given a* tile *as input, decides if the* tile *can be used* to *tile the entire plane. If the Heesch number is in fact bounded, this gives a simple algorithm for deciding if the tile can be used* to *tile the plane* [8]. The domino problem has a deep connection with the "Einstein" problem, which reads: Is there a single nonperiodic tile. *The nonexistence of a single nonperiodic tile* would imply *the existence of a decision method for the domino problem* [8].

A lot of geometric concepts could be illustrated in this context which should convince the students, among other, *that the good designer has to possess both the creative imagination of an artist and profound mathematical knowledge* [6].

These are only few examples where the tessellation could be used for introducing notions at school level. The tessellation is a "golden mine" and we can "exploit" it as much as we want pursuing our goal – to show to people that everyone can understand mathematics but the ways to achieve that are different.

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МАТЕМАТИКАТА ОТЕГЧАВА? КАКВО МОЖЕМ ДА НАПРАВИМ? ПАРКЕТИРАНЕТО КАТО ЕДНО РЕШЕНИЕ

Славица Грковска

В статията се разглежда как въвеждането на проблема за паркетирането на равнината в часовете по математика може да мотивира учениците да усвоят по-добре такива важни математически факти и понятия, каквито са свойствата на правилните многоъгълници, геометричните трансформации в равнината и др. Паркетирането на равнината е представено като подходящ начин тези понятия да се въведат както в традиционните часове по математика, така и в контекста на приложение на новите информационни технологии.