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ON BINDING-HOSKINS-PONZO TYPE INEQUALITIES*

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In this paper we adapt the techniques in [1] to obtain the best possible bounds on the zeros of the characteristic equation of a positive definite matrix, obtained from the Weierstrass series, when the determinant D, trace T and stopping criterion q for the Weierstrass iteration method are given.
Key words: Binding-Hoskins-Ponzo inequality, eigenvalues, upper and lower bounds, Weierstrass series, stopping criterion.

1. Introduction. Let A be a positive definite matrix, and let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, $\lambda_i > 0$, $i = 1, 2, \ldots, n$ be the distinct zeros of the characteristic equation (of A)

$$\det(B - \lambda I) = f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0. \tag{1}$$

The following theorem is very often applicable

Theorem A (Binding, Hoskins, Ponzo [1]). *The smallest and largest of the two positive real roots of*

$$t = T - (n-1)\left(\frac{D}{t}\right)^{\frac{1}{n-1}},$$

or

$$t = D\left(\frac{n-1}{T-t}\right)^{n-1}$$

provide, respectively, the greatest lower bound and least upper bound of the eigenvalues of the positive $n \times n$ matrix A, where $D = \det(A)$ and $T = trace(A)$, i.e.

$$\max_i \lambda_i \le t_2; \;\; t_1 \le \min_i \lambda_i.$$

In practice, the popular formulae for the simultaneous approximation of the eigenvalues (the roots of the equation (1)) is the Weierstrass procedure

$$\lambda_i^{k+1} = \lambda_i^k - \frac{f(\lambda_i^k)}{\prod\limits_{j\neq i}^{n}(\lambda_i^k - \lambda_j^k)}, \;\; i = 1, 2, \ldots, n; \;\; k = 0, 1, 2, \ldots. \tag{2}$$

The great importance for the user is the knowledge of upper and lower bounds on the approximations λ_i^k.

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2. Main results. Let

$$0 < x \le \lambda_1^s \le \lambda_2^s \le \cdots \le \lambda_{n-2}^s \le y, \;\; s = 0, 1, 2, \ldots$$

be the approximations, obtained from the Weierstrass series. For λ_i^k, λ_i^{k+1}, determined by (2), it holds, [2] :

$$\sum_{i=1}^{n} \lambda_i^{k+1} = -a_{n-1},$$

(3)

$$\sum_{i=1}^{n} \lambda_i^{k+1} \prod_{j \ne i}^{n} \lambda_j^k - (n-1) \prod_{j=1}^{n} \lambda_j^k = (-1)^n a_0.$$

However, $\lambda_i^{k+1} = q_i \lambda_i^k$, where $\sum_{i=1}^{n} q_i = q$ is an absolute constant for the method (2) (stopping criterion) and

$$-a_{n-1} = \sum_{i=1}^{n} \lambda_i = T,$$

$$(-1)^n a_0 = \prod_{i=1}^{n} \lambda_i = D.$$

Evidently,

$$x + \sum_{i=1}^{n-2} \lambda_i^k + y = T,$$

$$\frac{\lambda_1^{k+1}}{\lambda_1^k} \prod_{j=1}^{n} \lambda_j^k + \cdots + \frac{\lambda_n^{k+1}}{\lambda_n^k} \prod_{j=1}^{n} \lambda_j^k - (n-1) \prod_{j=1}^{n} \lambda_j^k = D,$$

$$xy \prod_{j=1}^{n-2} \lambda_j^k \left(\sum_{i=1}^{n} \frac{\lambda_i^{k+1}}{\lambda_i^k} - (n-1) \right) = D,$$

$$xy \prod_{j=1}^{n-2} \lambda_j^k (q + 1 - n) = D,$$

and

(4) $$(n-1)x + y \le T \le x + (n-1)y,$$

(5) $$(1 + q - n)x^{n-1} y \le D \le (1 + q - n) x y^{n-1}.$$

Note that $\dfrac{T - x - y}{n-2}$ = arithmetic mean of the $\lambda_i^k = \alpha$ and $\left(\dfrac{D}{(1+q-n)xy} \right)^{\frac{1}{n-2}} =$ geometric mean of the $\lambda_i^k = \beta$. Evidently, $\alpha \ge \beta$ for positive λ_i^k and

$$\frac{T - x - y}{n-2} \ge \left(\frac{D}{(1+q-n)xy} \right)^{\frac{1}{n-2}}.$$

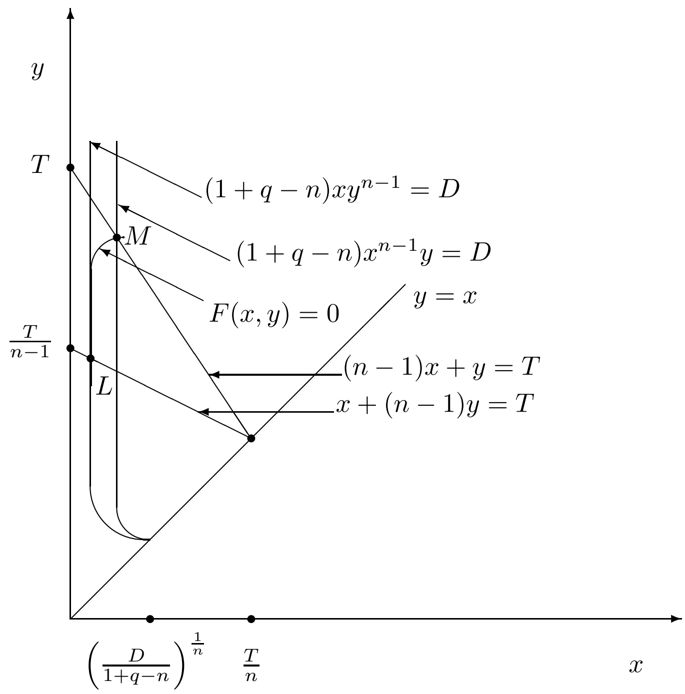

Fig. 1 Adapted construction by Binding, Hoskins and Ponzo

The contour (see, Fig.1)

$$F(x,y) = (1+q-n)xy(T-x-y)^{n-2} - D(n-2)^{n-2} = 0$$

is easily shown to pass through $L(x_L, y_L)$ and $M(x_M, y_M)$ the points of intersection of the bounding contours described by (4) and (5).

Note that

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\left(\frac{y}{x}\right)\frac{T-((n-1)x+y)}{T-(x+(n-1)y)} = \begin{cases} \infty & at\ L \\ 0 & at\ M \end{cases},$$

x_L and y_M are respectively the smaller and larger of the two positive real roots of

$$z = T - (n-1)\left(\frac{D}{(1+q-n)z}\right)^{\frac{1}{n-1}}, \tag{6}$$

which may also be written

$$z = \frac{D}{1+q-n}\left(\frac{n-1}{T-z}\right)^{n-1}. \tag{7}$$

We summarize the above

Theorem B. *The smallest and largest of the two positive real roots of (6) (or (7)) provide, respectively, the greatest lower bound and least upper bound of the approximations* λ_i^k*, when* $D = (-1)^n a_0$*,* $T = -a_{n-1}$ *and stopping criterion* q *are given, i.e.*

$$\max_i \lambda_i^k \leq z_2; \;\; z_1 \leq \min_i \lambda_i^k.$$

As an immediate consequence of Theorem B, we have the following corollary.

Corollary. *Each member of the sequence* $\{x_i\}_0^\infty$*,*

$$x_{i+1} = \frac{D}{1+q-n}\left(\frac{n-1}{T-x_i}\right)^{n-1}, \;\; x_0 = 0,$$

provides a lower bound on the λ_j^k *and* $\lim x_i = x_L$*.*

Each member of the sequence $\{y_i\}_0^\infty$*,*

$$y_{i+1} = T - (n-1)\left(\frac{D}{(1+q-n)y_i}\right)^{\frac{1}{n-1}}, \;\; y_0 = T,$$

provides a upper bound on the λ_j^k *and* $\lim y_i = y_M$*.*

Remark 1. In case when approximations λ_i^k, obtained from the Weierstrass series are equal to zeros of the characteristic equation (1) - λ_i (i.e. $\sum_{i=1}^{n} q_i = q = n$), we have the result due to Binding, Hoskins and Ponzo.

Remark 2. Let λ_i are located in n non-intersecting intervals $L_i^0 = [\underline{\lambda}_i^0, \overline{\lambda}_i^0]$, $i = 1, \ldots, n$, that is $L_i^0 \bigcap L_j^0 = \emptyset$ for $i \neq j$ and $\lambda \in L_i^0$ for $i = 1, \ldots, n$. The two-sided Weierstrass method can be written as

$$\overline{\lambda}_i^{k+1} = \overline{\lambda}_i^k - \frac{f(\overline{\lambda}_i^k)}{\prod_{j=1}^{i-1}(\overline{\lambda}_i^k - \underline{\lambda}_j^k)\prod_{j=i+1}^{n}(\overline{\lambda}_i^k - \overline{\lambda}_j^k)}$$

$$\underline{\lambda}_i^{k+1} = \underline{\lambda}_i^k - \frac{f(\underline{\lambda}_i^k)}{\prod_{j=1}^{i-1}(\underline{\lambda}_i^k - \underline{\lambda}_j^k)\prod_{j=i+1}^{n}(\underline{\lambda}_i^k - \overline{\lambda}_j^k)},$$

$$i = 1, \ldots, n; \; k = 0, 1, \ldots.$$

Then the estimations of the λ_i^k can be improved using the approach given in this paper, and by using the explicit formulae [2]

$$\sum_{i=1}^{n}\overline{\lambda}_i^{k+1}\prod_{j=1}^{i-1}\frac{\overline{\lambda}_i^k - \underline{\lambda}_j^k}{\overline{\lambda}_i^k - \overline{\lambda}_j^k} = \sum_{i=1}^{n}\overline{\lambda}_i^k\left(\prod_{j=1}^{i-1}\frac{\overline{\lambda}_i^k - \underline{\lambda}_j^k}{\overline{\lambda}_i^k - \overline{\lambda}_j^k} - 1\right) - a_{n-1},$$

$$\sum_{i=1}^{n}\overline{\lambda}_i^{k+1}\prod_{j\neq i}^{n}\overline{\lambda}_j^k\prod_{j=1}^{i-1}\frac{\overline{\lambda}_i^k - \underline{\lambda}_j^k}{\overline{\lambda}_i^k - \overline{\lambda}_j^k} = \sum_{i=1}^{n}\overline{\lambda}_i^k\prod_{j\neq i}^{n}\overline{\lambda}_j^k\left(\prod_{j=1}^{i-1}\frac{\overline{\lambda}_i^k - \underline{\lambda}_j^k}{\overline{\lambda}_i^k - \overline{\lambda}_j^k} - 1\right) + (n-1)\prod_{j=1}^{n}\overline{\lambda}_j^k + (-1)^n a_0,$$

or

$$\sum_{i=1}^{n}\underline{\lambda}_i^{k+1}\prod_{j=1}^{i-1}\frac{\underline{\lambda}_i^k-\overline{\lambda}_j^k}{\underline{\lambda}_i^k-\underline{\lambda}_j^k} = \sum_{i=1}^{n}\underline{\lambda}_i^k\left(\prod_{j=1}^{i-1}\frac{\underline{\lambda}_i^k-\overline{\lambda}_j^k}{\underline{\lambda}_i^k-\underline{\lambda}_j^k}-1\right)-a_{n-1},$$

$$\sum_{i=1}^{n}\underline{\lambda}_i^{k+1}\prod_{j\neq i}^{n}\underline{\lambda}_j^k\prod_{j=1}^{i-1}\frac{\underline{\lambda}_i^k-\overline{\lambda}_j^k}{\underline{\lambda}_i^k-\underline{\lambda}_j^k} = \sum_{i=1}^{n}\underline{\lambda}_i^k\prod_{j\neq i}^{n}\underline{\lambda}_j^k\left(\prod_{j=1}^{i-1}\frac{\underline{\lambda}_i^k-\overline{\lambda}_j^k}{\underline{\lambda}_i^k-\underline{\lambda}_j^k}-1\right)+(n-1)\prod_{j=1}^{n}\underline{\lambda}_j^k+(-1)^n a_0.$$

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ВЪРХУ НЕРАВЕНСТВА ОТ ТИП БИНДИНГ-ХОСКИНС-ПОНЗО

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В тази работа са получени оценки за корените на характеристично уравнение на положително дефинитна матрица.