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A CRITICAL BRANCHING PROCESS WITH INCREASING OFFSPRING VARIANCE*

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We study critical Bienaymé-Galton-Watson branching processes with increasing offspring variance in the particular case of a geometric offspring distribution. The increasing variance has a decreasing effect on the process in the sense that the rate of the non-extinction probability decreases. We obtain limit theorems for the process subject to either state-independent or state-dependent immigration.

1. Introduction. The Bienaymé-Galton-Watson branching process $\{\mu_n\}$ can be defined by the recurrence formula

$$\mu_n = \sum_{i=1}^{\mu_{n-1}} X_i(n), \qquad n = 1, 2, \ldots; \quad \mu_0 \equiv 1, \tag{1}$$

where $\{X_i(n)\},\ i, n = 1, 2, \ldots$ are independent and identically distributed random variables, taking on nonnegative integer values.

We are interested in branching processes with varying offspring variance. These models can be considered within the framework of branching processes with varying environments (see e.g. Jagers(1974) [2]). In particular we want to study the case when the offspring mean is fixed but the offspring variance increases to infinity along with the generation index (time). Although all three classes processes: subcritical, critical, and subcritical are of interest, here we shell focus on the critical case. The behavior of the critical processes is of particular interest from analytical viewpoint (obtaining new limit theorems) as well as from modelling perspective, to see whether the process can reach a stationary state. As we see below, the increasing offspring variance has a decreasing effect to the population in the sense of decreasing the rate of the non-extinction probability. Therefore, it is worth considering an immigration component that brings "immigrants" from an outside source and balances the population.

Further on we shell study the particular case of a geometric offspring distribution with mean one given by

$$p_n(k) = P(\mu_n = k | \mu_{n-1} = 1), \tag{2}$$

where for $0 \le p_n < 1$,

$$p_n(0) = p_n \qquad \text{and} \qquad p_n(k) = (1 - p_n)^2 p_n^{k-1}, \qquad k = 1, 2, \ldots$$

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The offspring mean and variance are

$$EX_i(n) = 1, \qquad VarX_i(n) = \frac{2p_n}{1-p_n} = 2b_n < \infty, \quad \text{say.}$$

The offspring pgf is given by

$$f_n(s) = \sum_{k=0}^{\infty} p_n(k)s^k = \frac{p_n - (2p_n - 1)s}{1 - p_n s} = \frac{b_n(1-s)+s}{b_n(1-s)+1}$$

After discussing the process $\{\mu_n\}$ (without immigration) in Section 2, we present some results for $\{\mu_n\}$ subject two different immigration policies: state-independent (Section 3) and state-dependent (Section 4) immigration. The last section contains some concluding remarks.

2. Processes without Immigration. Let $g_n(s)$ be the pgf of μ_n assuming a geometric offspring distribution defined by (2). Hence,

$$g_n(s) = \sum_{k=0}^{\infty} P(\mu_n = k)s^k = \frac{\sigma_n(1-s)+s}{\sigma_n(1-s)+1},$$

where $\sigma_n = \sum_{k=1}^{n} b_k$.

From here it is not difficult to see that $E\mu_n = 1$, $Var\mu_n = 2\sigma_n$ and

$$P(\mu_n > 0) = 1 - g_n(0) = \frac{1}{\sigma_n + 1}.$$

Therefore, if $\inf_n b_n > 0$ then

$$\lim_{n\to\infty} P(\mu_n = 0) = 1.$$

Note that, if $b_n \to \infty$, then $P(\mu_n = 0) \to 1$ faster than in a process with constant offspring variance, whereas $b_n \to b < \infty$ results in the same rate $1/(bn)$ for processes with both constant and varying offspring variance.

As a direct corollary of Theorem 5 in Jagers(1974) [2] we have

Theorem 1 (Jagers(1974) [2]). *Suppose that* $\inf_n b_n > 0$. *Then*

$$\lim_{n\to\infty} P(\mu_n/\sigma_n > x | \mu_n > 0) = e^{-x}, \qquad x \geq 0$$

where $\sigma_n = \sum_{k=1}^{n} b_k$.

Comment. This result shows that despite the fact that $P(\mu_n = 0) \to 1$ faster than when the variance is constant, the quasi-stationary distribution is still exponential. Notice that, the offspring variance increases to infinity as $n \to \infty$ remaining finite for any fixed n.

3. Processes with State-Independent Immigration. Consider a branching process with immigration defined by the recurrence

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n) + Y_n, \qquad n = 1, 2, \ldots; \quad Z_0 \equiv 1,$$

where Z_n represents the population size at time n of the process $\{\mu_n\}$ in (1) that has been modified to allow immigration. Y_n particles enter the system at time n, where the random variables $\{Y_n\}$ are independent and identically distributed with pgf

$$h(s) = 1 - \alpha(1-s) + \beta(1-s)^2$$

Let us assume geometric offspring (2) with increasing variance, i.e.,

$$\lim_{n\to\infty} b_n \uparrow \infty.$$

For the process' pgf we have

$$\begin{aligned} F_n(s) &= \prod_{k=0}^{n} h(g_{n-k}(s)) \\ &= \prod_{k=0}^{n} \left(1 - \frac{\alpha(1-s)}{\sigma_{n-k}(1-s)+1} + \frac{\beta(1-s)^2}{(\sigma_{n-k}(1-s)+1)^2}\right) \end{aligned} \tag{3}$$

Set $s = 0$ in the above equation. Then

$$\begin{aligned} P(Z_n = 0) &= \prod_{k=0}^{n} \left(1 - \frac{\alpha}{\sigma_{n-k}+1} + \frac{\beta}{(\sigma_{n-k}+1)^2}\right) \\ &= \exp\left\{\sum_{k=0}^{n} \log\left(1 - \frac{\alpha}{\sigma_{n-k}+1} + \frac{\beta}{(\sigma_{n-k}+1)^2}\right)\right\} \\ &\sim \exp\left\{-\sum_{k=0}^{n} \left(\frac{\alpha}{\sigma_{n-k}+1} - \frac{\beta}{(\sigma_{n-k}+1)^2}\right)\right\}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P(Z_n = 0) = \begin{cases} \exp\left\{-\sum_{n=0}^{\infty} \left(\frac{\alpha}{\sigma_n+1} - \frac{\beta}{(\sigma_n+1)^2}\right)\right\} & \text{if } \sum_{n=0}^{\infty} 1/\sigma_n < \infty, \\ 0 & \text{if } \sum_{n=0}^{\infty} 1/\sigma_n = \infty. \end{cases}$$

Note that, the probability for extinction would be smaller if either the immigration mean is bigger or the offspring variance goes to infinity in a slower rate.

Using a standard argument based on the particular form (3) of process' pgf we obtain

Theorem 2.

$$\lim_{n\to\infty} P(Z_n = k) = \begin{cases} r_k & \text{if } \sum_{n=0}^{\infty} 1/\sigma_n < \infty \\ 0 & \text{if } \sum_{n=0}^{\infty} 1/\sigma_n = \infty, \end{cases}$$

where $\sigma_n = \sum_{k=1}^{n} b_k$ *and* $\{r_k\}, k = 0, 1, \ldots$ *is a probability distribution with pgf*

$$R(s) = \exp\left\{-\sum_{n}^{\infty} \left(\frac{\alpha(1-s)}{1+\sigma_n(1-s)} - \frac{\beta(1-s)^2}{(1+\sigma_n(1-s))^2}\right)\right\}.$$

Comment. It is clear that if the offspring variance increases sufficiently fast, then the series above will be convergent and the process has a non-degenerate limiting distribution.

4. Processes with State-Dependent Immigration. Consider, on a probability space (Ω, A, P), a sequence of independent Bienaymé-Galton-Watson branching processes $\{\mu_n(k)\}$, $k = 1, 2, \ldots$ with identical geometric offspring distributions (2). Denote by T_k

the time to extinction of $\mu_n(k)$, i.e., for any n

$$P(T_k > n) = P(\mu_n(k) > 0)$$

Let us construct a discrete renewal process $\{S_n\}$ $n = 1, 2, \ldots$ as follows: $S_0 = 0$,

$$S_{k+1} = S_k + T_{k+1}, \qquad k \geq 0$$

and set

$$N(n) = \max\{k : \ S_k \leq n\} \ .$$

Then the well-known Bienaymé-Galton-Watson branching process with immigration at zero only $\{Z_n^0\}$ can be defined by

$$Z_n^0 = \mu_{n-S_{N(n)}}(N(n)+1)$$

Note that, $\{Z_n^0\}$ is a renewal process and therefore its asymptotic behavior depends on whether its mean renewal time ET_k is finite or not. Recall that $\sigma_n = \sum_{k=1}^{n} b_k$.

Case 1. $\sum_{n=1}^{\infty} 1/\sigma_n < \infty$. Since

$$P(\mu_n(k) > 0) = \frac{1}{\sigma_n + 1}$$

then

$$ET_k = \sum_{n=0}^{\infty} P(T_k > n) = \sum_{n=o}^{\infty} P(\mu_n(k) > 0) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n + 1} < \infty \tag{4}$$

The fact that the mean renewal time is finite, allows us to apply well-known classical renewal theory results (see Feller (1966) Ch. XI. 8, 365–366) and obtain

Theorem 3. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n} < \infty$$

then for $\{Z_n^0\}$ there exists a proper non-degenerate limiting distribution given by

$$\lim_{n\to\infty} P(Z_n^0 = k) = \frac{1}{ET_k} \sum_{j=0}^{\infty} P(\mu_j(1) = k, \ T_1 > j),$$

where $ET_k, \ k = 0, 1, \ldots$ are from (4).

Case 2. $\sum_{n=1}^{\infty} 1/\sigma_n = \infty$. Note that now the mean renewal time is infinite. Let us assume that as $k \to \infty$

$$b_k = L(k) \uparrow \infty,$$

where $L(x)$ is a slowly varying at infinity function (sfv). Then

$$\sigma_n = \sum_{k=0}^{n} b_k \sim nL(n)$$

and (see Feller (1966) [1], Ch. VIII. 9, 272–273)

$$\sum_{k=0}^{n} \frac{1}{\sigma_k + 1} = L^*(n)$$

where $L^*(n)$ is an increasing svf, such that $1/L(n) = o(L^*(n))$.

We are in a position to state

Theorem 4. *If $b_n = L(n) \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{1}{\sigma_n} = \infty$$

then for $0 < x < 1$

$$\lim_{n\to\infty} P\left(\frac{L^*(M(Z_n^0))}{L^*(n)} < x\right) = x \ ,$$

where $M(\cdot)$ is the inverse function of $xL(x)$ for $x > 0$.

The proof of Theorem 4 follows closely that of Theorem 5.2 (ii) in Mitov (1999) [3] and it is omitted here.

Example 1. Let $b_k = \log^\beta k$, where $0 < \beta < 1$. Then $\sigma_n \sim n \log^\beta n$ and $L^*(n) = \log^{1-\beta} n/(1-\beta)$. In addition, the function $x \log^\beta x$ is increasing on $[1,\infty)$ and has an inverse $M_\beta(x)$, say. Therefore, Theorem 4 leads to

$$\lim_{n\to\infty} P\left(\frac{\log^{1-\beta}\left(M_\beta(Z_n^0)\right)}{\log^{1-\beta}(n)} < x\right) = x \ ,$$

for $0 < x < 1$.

Example 2. Let $b_k = \log k$. Then $\sigma_n \sim n \log n$ and $L^*(n) = \log\log n$. Let $M_1(x)$ be the inverse of $x \log x$. Now, Theorem 4 implies for $0 < x < 1$

$$\lim_{n\to\infty} P\left(\frac{\log\log M_1(Z_n^0)}{\log\log n} < x\right) = x \ .$$

5. Concluding remarks. We would like to point out the intermediate position of the results between those in the finite and infinite variance cases. It is clear that the rate at which the offspring variance increases pulls the asymptotic behavior of the process to either one of the above well-studied cases. More precisely, if $\sum_{n=0}^{\infty} P(\mu_n(k) > 0) < \infty$ then the results are similar to those in the finite variance case. If the offspring variance increases faster, such that $\sum_{n=0}^{\infty} P(\mu_n(k) > 0) = \infty$, then the results are similar to the ones when the variance is infinite.

In the critical case the assumption for an increasing variance can be stated in a more general setting considering an offspring pgf of the form

$$f_n(s) = s + (1-s)^{1+\alpha} L_n\left(\frac{1}{1-s}\right), \qquad 0 < \alpha \le 1 \tag{5}$$

where $\{L_n(x)\}$ is a sequence of slowly varying at infinity functions (sfv) in x. Note that $f_n(s)$ can be represented in the form (5) as

$$f_n(s) = s + \frac{1}{1-s+b_n^{-1}}(1-s)^2$$

and

$$\lim_{s\to 1} \frac{1}{1-s+b_n^{-1}} = b_n < \infty \ .$$

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КРИТИЧЕСКИ РАЗКЛОНЯВАЩ СЕ ПРОЦЕС С РАСТЯЩА ДИСПЕРСИЯ НА ПОТОМСТВОТО

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Разглежда се критически процес на Галтон-Уотсън с растяща дисперсия на потомството, в частния случай, когато разпределението на потомците е геометрично. Оказва се, че с увеличаването на дисперсията вероятността за неизраждане намалява по-бързо с времето. Получени са гранични теореми за процеса при допускане на обща имиграция и на имиграция, зависеща от състоянието.