МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2002
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2002
Proceedings of Thirty First Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 3–6, 2002

I-FIXED NOT I-CRITICAL VERTICES*

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If a vertex x of a graph G is i-fixed and incident only to bridges, then x is i-critical.

1. Introduction. For a graph theory terminology not presented here we follow [2]. All graphs discussed here are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. If $X \subseteq V(G)$ then $\langle X, G\rangle$ is the induced subgraph of G with the vertex set X. For a vertex v of G its neighborhood $N(v,G)$ is $\{x \in V(G) | vx \in E(G)\}$ and its closed neighborhood $N[v,G]$ is $N(v,G) \cup \{v\}$. For a set $S \subseteq V(G)$, its open neighborhood $N(S,G)$ is the set of all vertices adjacent to any vertex in S, and its closed neighborhood $N[S,G]$ is $N(S,G) \cup S$. A set $S \subseteq V(G)$ is a *dominating set* of a graph G if $N[S,G] = V(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A dominating set S is an *independent dominating set* (*id*-set) if no two vertices in S are adjacent, that is, S is an independent set. The *independent domination number* $i(G)$ of a graph G is the minimum cardinality of an *id*-set of G. A set S of vertices in a graph G is a *vertex neighborhood set* (n-set) if $G = \bigcup_{v \in S} \langle N[v,G], G\rangle$. The *vertex neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set.

We shall employ e to represent an element, either a vertex or an edge, of a graph G and t to be the cardinality of a set of elements with some prescribed property. Then a t-set is a set with that property. Following *Sampathkumar* ([2], pp. 291), an element e of G is:

(i) *t-critical* if $t(G-e) \neq t(G)$;
(ii) *t^+-critical* if $t(G-e) > t(G)$;
(iii) *t^--critical* if $t(G-e) < t(G)$;
(iv) *t-fixed* if e belongs to every t-set;
(v) *t-free* if e belongs to some t-set but not to all t-sets;
(vi) *t-totally free* if e belongs to no t-set.

Sampathkumar and Neerlagi [3] have studied the relationship among such types of elements when $t = \gamma$ or $t = n_0$. Here we shall be concerned with the independent domination number. For a graph G we define:

$\mathcal{I}(G)$ - the set of all i-sets of G;
$I(G) = \{x \in V(G) | x$ belongs to some i-set of $G\}$.

*Math. Subject Classification: 05C69

$IN(G) = \{x \in V(G) | x \text{ is } i \text{ - totally free }\}$;
$IK(G) = \{x \in V(G) | x \text{ is } i\text{-fixed }\}$;
$I_0(G) = \{x \in V(G) | x \text{ is } i\text{-free and } i(G-x) = i(G)\}$;
$I_{-1}(G) = \{x \in V(G) | x \text{ is } i\text{-free and } i(G-x) = i(G) - 1\}$;
$IK_p(G) = \begin{cases} \{x \in V(G) | i(G-x) = i(G) + p\} & \text{for} \quad p \geq 1 \\ \{x \in IK(G) | i(G-x) = i(G) + p\} & \text{for} \quad p \leq 0 \end{cases}$

We shall begin with two lemmas.

Lemma A ([2]). *Let G be a graph of an order at least two and $x \in V(G)$. Then $i(G) - 1 \leq i(G-x) \leq |V(G)| - 1$.*

Lemma 1. *Let G be a graph with $n \geq 2$ vertices. Then $IK(G) = \bigcup_{p=-1}^{n-2} IK_p(G)$.*

Proof. Let $x \in V(G)$. Then by Lemma A, it follows that:

$$i(G-x) = i(G) + r \text{ for some } r, -1 \leq r \leq n-2. \tag{1}$$

If $x \in IK(G)$, then by (1) $x \in \bigcup_{p=-1}^{n-2} IK_p(G)$. Let now $x \in IK_p(G)$. Hence by (1) : $-1 \leq p \leq n-2$. Let $p \geq 1$. Then every *id*-set M of G such that $x \notin M$ is an *id*-set of $G-x$. Hence $|M| \geq i(G-x) > i(G)$ and then x is i-fixed.

By the above definitions and lemmas we have:

Proposition 2. *For a graph G of an order at least two:*
1) $I(G) = I_{-1}(G) \cup I_0(G) \cup IK(G)$.
2) $\{x \in V(G) | i(G-x) = i(G)\} = IN(G) \cup I_0(G) \cup IK_0(G)$.
3) $\{x \in V(G) | x$ *is i-free* $\} = I_{-1}(G) \cup I_0(G)$.
4) $\{x \in V(G) | x$ *is i^--critical* $\} = I_{-1}(G) \cup IK_{-1}(G)$.

Proposition 3. *Let G be a graph, $|V(G)| \geq 2$ and $x \in V(G)$.*
1) *x is i-fixed if and only if $N(x,G) \subseteq IN(G)$.*
2) *Let x be i^--critical. Then $N(x,G) \subseteq IN(G-x)$.*
3) *Let $x \in IK_0(G)$. Then x is incident to no pendant edge. If $Q \in \mathcal{I}(G-x)$ then $Q \cup \{x\}$ is an id-set of G.*

Proof. 1) Let x be i-fixed. Then for every $M \in \mathcal{I}(G) : M \cap N(x,G) = \emptyset$. Hence $N(x,G) \subseteq IN(G)$.

Let $N(x,G) \subseteq IN(G)$. Then x belongs to every i-set. Hence x is i-fixed.

2) Let $M \in \mathcal{I}(G-x)$. Then $|M| = i(G) - 1$ and therefore $M \cap N(x,G) = \emptyset$. Hence $N(x,G) \subseteq IN(G-x)$.

3) Let $xy \in E(G)$. Suppose $N(y,G) = \{x\}$. Let $M \in \mathcal{I}(G-x)$. Hence $y \in M$ and then M is an *id*-set of G with $|M| = i(G-x) = i(G)$. Therefore $M \in \mathcal{I}(G)$ and $x \notin M$ – a contradiction. So, $deg(y,G) > 1$.

Suppose $N(x,G) = \{y\}$. Let $M \in \mathcal{I}(G)$. Then $x \in M, y \notin M$ and $|M| > 1$. Hence $N[M - \{x\}, G] = V(G) - \{x,y\}$ - otherwise x is i^--critical. But then $S = (M - \{x\}) \cup \{y\} \in \mathcal{I}(G)$ and $x \notin S$ - a contradiction. So, $deg(x,G) > 1$.

Since $x \in IK_0(G)$, if $Q \in \mathcal{I}(G-x)$ then $|Q| = i(G-x) = i(G)$ and $M \cap N(x,G) = \emptyset$. Hence $Q \cup \{x\}$ is an *id*-set of G.

Theorem 4. *Let G_1 and G_2 be vertex disjoint graphs, $x_j \in V(G_j)$, $j = 1, 2$ and $G = (G_1 \cup G_2) + x_1x_2$. If $x_1 \in IK_0(G)$, then $x_1 \in IK_0(G_1)$ and $x_2 \in IN(G) \cap IN(G_2)$.*

Proof. Let $x_1 \in IK_0(G)$ and $i(x_1, G) = min\{|M||M$ is an *id*-set of G and $x_1 \in M\}$. By Proposition 3: $x_2 \in IN(G)$ and $deg(x_j, G) > 1$ for $j = 1, 2$. Then:

$$i(G) = i(G - x_1) = i(G_1 - x_1) + i(G_2) \tag{2}$$

and

$$i(G) = i(x_1, G_1) + i(G_2 - x_2). \tag{3}$$

Suppose $x_1 \in I_0(G_1) \cup IN(G_1)$. Then by (2): $i(G) = i(G_1) + i(G_2)$. Let $x_1 \notin M_1 \in \mathcal{I}(G_1)$ and $M_2 \in \mathcal{I}(G_2)$. Then $M = M_1 \cup M_2$ is an *id*-set of G and $|M| = i(G_1) + i(G_2) = i(G)$. Hence $M \in \mathcal{I}(G)$ and $x_1 \notin M$ – a contradiction.

Suppose $x_1 \in I_{-1}(G_1) \cup IK_{-1}(G_1)$. Then by (2) and (3): $i(G) = i(G_1) + i(G_2) - 1$ and $i(G) = i(G_1) + i(G_2 - x_2)$. Hence $i(G_2 - x_2) = i(G_2) - 1$ and then $x_2 \in I_{-1}(G_2) \cup IK_{-1}(G_2)$. Let $M_j \in \mathcal{I}(G_j - x_j), j = 1, 2$. By Proposition 3 the set $L = M_1 \cup M_2 \cup \{x_2\}$ is an *id*-set of G with $|L| = i(G)$. Hence $L \in \mathcal{I}(G)$ and $x_1 \notin L$ – a contradiction.

So, $x_1 \in IK_p(G_1)$ for some $p \geq 0$. Now, by (2) and (3) it follows that $i(G) = i(G_1) + i(G_2) + p$ and $i(G) = i(G_1) + i(G_2 - x_2)$. Hence $i(G_2 - x_2) = i(G_2) + p$. Suppose $x_2 \notin IN(G_2)$. Let then $x_2 \in M_2 \in \mathcal{I}(G_2)$ and $M_1 \in \mathcal{I}(G_1 - x_1)$. Hence $M = M_1 \cup M_2$ is an *id*-set of G with $|M| = i(G_1) + p + i(G_2) = i(G)$ and then $M \in \mathcal{I}(G)$. But $x_1 \notin M$ - a contradiction. Hence $x_2 \in IN(G_2)$ and then $i(G_2 - x_2) = i(G_2)$. So $p = 0$ and we have the result.

Corollary 5. *Let a vertex x of a graph G be incident only to bridges. Then $x \notin IK_0(G)$.*

Proof. Suppose $x \in IK_0(G)$. Let $e_1, \ldots, e_k$ be the edges which are incident to x. Because of Proposition 3 - $k > 1$. Let $G = H_0$ and H_i be the component of $H_{i-1} - e_i$ which contains x, where $i = 1, \ldots, k-1$. By Theorem 4 we have that $x \in IK_0(H_i)$ for $i = 1, \ldots, k-1$.Since x is an endvertex of H_{k-1}, by Proposition 3 follows that $x \notin IK_0(H_{k-1})$. So we have a contradiction.

Corollary 6. *For every tree T of order $n \geq 2$: $IK_0(T) = \emptyset$.*

Example. For $n \geq 12$, let S_n be a graph defined as follows: $V(S_n) = \{a, b, c, i_1, \ldots, i_7, j_{11}, \ldots, j_n\}$ and $E(S_n) = \{i_1i_2, i_1i_3, ai_2, \ldots, ai_5, i_4i_7, i_5i_7, bi_6, bi_7, i_7j_{11}, cj_{11}, \ldots, cj_n\}$. It is easy to see that a cutvertex $a \in IK_0(S_n)$.

Counterexample. In [1] (see also [2] pp.292), it is claimed that if a cutvertex x of G is γ-fixed then x is γ^+-critical. Note that this is false. It is easy to see that for a graph S_n, $n \geq 12$, the cutvertex a is γ-fixed and $\gamma(G - a) = \gamma(G)$.

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I-ФИКСИРАНИ НЕ I-КРИТИЧНИ ВЪРХОВЕ

Владимир Димитров Самодивкин

Ако връх на граф е i-фиксиран и е инцидентен само с мостове, то той е i-критичен.