МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2002
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2002
*Proceedings of Thirty First Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 3–6, 2002*

THE OPTIMALITY ALTERNATIVES AT DECISION MAKING *

Zdravko Dimitrov Slavov

In the present work we study the concept of Pareto optimality at decision making in a society with finitely many individuals. We consider the preference relations of the individuals and three versions of the Pareto optimality alternatives – weak, strong and full.

1. Introduction. We consider a society with n individuals. Let I be a set of individuals and $|I| = n \geq 2$, let A be a set of alternatives and $|A| > 2$, let $R = \{R^k\}_{k=1}^n$ be a profile of individual preference relations on A and each $i_k \in I$ has binary relation R^k such that for all alternatives $x, y \in A$, there is xR^ky if and only if $i_k \in I$ preferences x by y. The set A can be finite or infinite. Any relation R^k is reflective (if $x \in A$, then xR^kx), transitive (if $x, y, z \in A$, xR^ky and yR^kz, then xR^kz) and complete (if $x, y \in A$, then xR^ky holds or yR^kx holds). We denote the asymmetric part of R^k by P^k (for $x, y \in A$, there is xP^ky if and only if xR^ky holds and yR^kx does not hold). For P^k of $i_k \in I$, there is xP^ky if and only if $i_k \in I$ strictly preferences x by y. The relation P^k of $i_k \in I$ is transitive. We denote the symmetric part of R^k by I^k (for $x, y \in A$, there is xI^ky if and only if xR^ky and yR^kx hold). For relation I^k of $i_k \in I$, there is xI^ky if and only if $i_k \in I$ indifferences for x and y. The relation I^k of $i_k \in I$ is reflective and transitive.

We will consider two examples.

Example 1. Let us consider an exchange system. This system consists of finitely many agents and they exchange goods between each other. The agents form a society and they are the individuals. Let each agent $i_k \in I$ has endowment $w^k \in \Re^m_+$ and $v = \sum\limits_{i=1}^n w^i \in \Re^m_{++}$. Here $H = \{x\left(x^1, x^2, \dots, x^n\right) \in \Re^{mn}_+ : \sum\limits_{i=1}^n x^i = v\}$ is a set of individually rational allocations, where agent $i_k \in I$ owns of the goods $x^k\left(x^k_1, x^k_2, \dots, x^k_m\right) \in \Re^m_+$, a number $x^k_j \geq 0$ shows the quantity of good $g_j \in G$ property of this agent. Thus H is a set of alternatives. Here we have two cases:

First, if the goods are perfecting divisible, then a set of alternatives is infinite;

Second, if the goods are not perfecting divisible, then a set of alternatives is finite.

Example 2. Let us consider a game model. It consists of finitely many players and they form a society. In this model we have *"player" = "individual"*. Let each player $i_k \in I$ has a set of strategies M_k. Here $M_1 \times M_2 \times \cdot \times M_n$ is a set of alternatives.

---

*Math. Subject Classification: 90B50, 91A35, 91B06, 91B14

2. Optimality alternatives. The set $R_k(x) = \{y \in A : yR^k x\}$ we will call the set of weakly preference of $i_k \in I$. The sets $R_k(x)$ and $\bigcap_{k=1}^{n} R_k(x)$ are nonempty subsets of A, $x \in R_k(x)$ for all $i_k \in I$, $x \in \bigcap_{k=1}^{n} R_k(x)$.

The set $P_k(x) = \{y \in A : yP^k x\}$ we will call the set of strict preference of $i_k \in I$. The sets $P_k(x)$ and $\bigcap_{k=1}^{n} P_k(x)$ can be empty, $x \notin P_k(x)$ for all $i_k \in I$, $x \notin \bigcap_{k=1}^{n} P_k(x)$.

The set $I_k(x) = \{y \in A : yI^k x\}$ we will call the set of indifference of $i_k \in I$. The sets $I_k(x)$ and $\bigcap_{k=1}^{n} I_k(x)$ are nonempty subsets of A, $x \in I_k(x)$ for all $i_k \in I$, $x \in \bigcap_{k=1}^{n} I_k(x)$.

Definition 1. *An alternative $y \in A$ weakly dominates an alternative $x \in A$ if and only if $yR^k x$ for all $i_k \in I$ and $y \neq x$. We call that the alternative $x \in A$ is weak optimality if and only if there does not exist $y \in A$ such that y weakly dominates x. The set of the weak optimality alternatives of A will be denoted by O_w.*

It is easy to show that if $x, y \in A$, then y weakly dominates x if and only if $y \in \left(\bigcap_{k=1}^{n} R_k(x)\right) \backslash \{x\}$.

Theorem 1. *Let $x \in A$, for weak optimality alternatives the following statements are equivalent:*

(a) $x \in O_w$;

(b) $\{y \in A : yR^k x$ for all $i_k \in I$ and $y \neq x\}$ is empty.

Proof. From Definition 1 it follows the proof of Theorem 1.

Theorem 2. *Let $x \in A$, $x \in O_w$ if and only if $\{x\} = \bigcap_{k=1}^{n} R_k(x)$.*

Proof. Let $x \in O_w$ therefore the set

$\{y \in A \quad : \quad yR^k x$ for all $i_k \in I$ and $y \neq x\}$

is empty. From $x \in \bigcap_{k=1}^{n} R_k(x)$ it follows $\{x\} = \bigcap_{k=1}^{n} R_k(x)$.

Conversely, let $\{x\} = \bigcap_{k=1}^{n} R_k(x)$. From $x \in \bigcap_{k=1}^{n} R_k(x)$ it follows that the set

$\{y \in A \quad : \quad yR^k x$ for all $i_k \in I$ and $y \neq x\}$

is empty. As a result we obtain $x \in O_w$.

Definition 2. *An alternative $y \in A$ strongly dominates an alternative $x \in A$ if and only if $yR^k x$ for all $i_k \in I$ and $yP^m x$ for some $i_m \in I$. We call that the alternative $x \in A$ is strong optimality if and only if there does not exist $y \in A$ such that y strongly dominates x. The set of the strong optimality alternatives of A will be denoted by O_s.*

It is easy to show that if $x, y \in A$, then y strongly dominates x if and only if $y \in \left(\bigcap_{k=1}^{n} R_k(x)\right) \cap \left(\bigcup_{k=1}^{n} P_k(x)\right)$.

Theorem 3. *Let $x \in A$, for strong optimality alternatives the following statements are equivalent:*

(a) $x \in O_s$;

(b) $\{y \in A : yR^k x$ for all $i_k \in I$ and $yP^m x$ for some $i_m \in I\}$ is empty.

Proof. From Definition 2 it follows the proof of Theorem 3.

Definition 3. *An alternative $y \in A$ fully dominates an allocation $x \in A$ if and only if $yP^k x$ for all $i_k \in I$. We call that the alternative $x \in A$ is full optimality if and only if there does not exist $y \in A$ such that y fully dominates x. The set of the full optimality alternatives of A will be denoted by O_f.*

It is easy to show that if $x, y \in A$, then y fully dominates x if and only if $y \in \bigcap_{k=1}^{n} P_k(x)$.

Theorem 4. *Let $x \in A$, for full optimality allocations the following statements are equivalent:*

(a) $x \in O_f$;

(b) $\{y \in A : yP^k x \text{ for all } i_k \in I\}$ is empty.

Proof. From Definition 3 it follows the proof of Theorem 4.

Theorem 5. *Let $x \in A$, $x \in O_f$ if and only if the set $\bigcap_{k=1}^{n} P_k(x)$ is empty.*

Proof. From Theorem 4 it follows the proof of Theorem 5.

3. Main results. We will consider some characteristics of the Pareto optimality alternatives.

Theorem 6. *(a) If $x \in O_w$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y = x$;*

(b) If $x \in O_s$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y \in O_s$ and $y \in \bigcap_{k=1}^{n} I_k(x)$;

(c) If $x \in O_f$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y \in O_f$ and $y \in \bigcup_{k=1}^{n} I_k(x)$.

Proof. (a) From $x \in O_w$ and Theorem 2 we have $\{x\} = \bigcap_{k=1}^{n} R_k(x)$. Thus, there is $y \in \bigcap_{k=1}^{n} R_k(x) = \{x\}$ therefore $y = x$.

(b) From $x \in O_s$ and Theorem 3 it follows the set $\{z \in A : zR^k x$ for all $i_k \in I$ and $zP^m x$ for some $i_m \in I\}$ is empty. We have $yR^k x$ for all $i_k \in I$, therefore the set $\{z \in A : zR^k y$ for all $i_k \in I$ and $zP^m y$ for some $i_m \in I\}$ is empty too. As a result we obtain $y \in O_s$.

Let us assume that $y \notin \bigcap_{k=1}^{n} I_k(x)$. Thus, from $y \in \bigcap_{k=1}^{n} R_k(x)$ and $y \notin \bigcap_{k=1}^{n} I_k(x)$ it follows there exists $i_m \in I$ such that $yP^m x$. This contradicts to $x \in O_s$, therefore $y \in \bigcap_{k=1}^{n} I_k(x)$.

(c) From $x \in O_f$ and Theorem 4 it follows the set $\{z \in A : zP^k x$ for all $i_k \in I\}$ is empty. We have $yR^k x$ for all $i_k \in I$, therefore the set $\{z \in A : zP^k y$ for all $i_k \in I\}$ is empty too. As a result we obtain $y \in O_f$.

Let us assume that $y \notin \bigcup_{k=1}^{n} I_k(x)$, i.e. $y \notin I_k(x)$ for all $i_k \in I$. Thus, from $y \in \bigcap_{k=1}^{n} R_k(x)$, i.e. $y \in R_k(x)$ for all $i_k \in I$ and $y \notin I_k(x)$ for all $i_k \in I$ it follows $yP^k x$ for all $i_k \in I$. This contradicts to $x \in O_f$, therefore $y \in \bigcup_{k=1}^{n} I_k(x)$.

Theorem 7. *If $x \in O_w$, then* $\{x\} = \bigcap_{k=1}^{n} I_k(x)$.

Proof. Let $x \in O_w$ and let us assume $\{x\} \neq \bigcap_{k=1}^{n} I_k(x)$. From $\{x\} \neq \bigcap_{k=1}^{n} I_k(x)$ it follows there exists $y \in A$ such that $y \in \bigcap_{k=1}^{n} I_k(x)$ and $x \neq y$. From $\bigcap_{k=1}^{n} I_k(x) \subset \bigcap_{k=1}^{n} R_k(x)$ we obtain $y \in \bigcap_{k=1}^{n} R_k(x)$ and $x \neq y$. This contradicts to $x \in O_w$.

Theorem 8. $O_w \subset O_s \subset O_f$.

Proof. First, let $x \in O_w$ and let us assume that $x \notin O_s$. Following there exists $y \in A$ such that $yR^k x$ for all $i_k \in I$ and $yP^m x$ for some $i_m \in I$. As a result we have $y \in \bigcap_{k=1}^{n} R_k(x) = \{x\}$, therefore $x = y$. This contradicts to $yP^m x$ therefore $x \in O_s$.

Second, let $x \in O_s$ and let us assume that $x \notin O_f$. Following there exists $y \in A$ such that $yP^k x$ for all $i_k \in I$. We obtain $yR^k x$ for all $i_k \in I$ and $yP^m x$ for some $i_m \in I$. This contradicts to $x \in O_s$ therefore $x \in O_f$.

Theorem 9. *For $x \in A$ the following statements hold:*

(a) $x \in O_w$ if and only if $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ and $\left|\bigcap_{k=1}^{n} R_k(x)\right| = 1$;

(b) $x \in O_s$ if and only if $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$;

(c) $x \in O_f$ if and only if $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$.

Proof. (a) Let $x \in O_w$. From Theorem 2 we have $\{x\} = \bigcap_{k=1}^{n} R_k(x)$. Thus, we have $x \in \bigcap_{k=1}^{n} I_k(x) \subset \bigcap_{k=1}^{n} R_k(x) = \{x\}$ therefore $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x) = \{x\}$. Finally, we obtain $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ and $\left|\bigcap_{k=1}^{n} R_k(x)\right| = 1$.

Conversely, let $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ and $\left|\bigcap_{k=1}^{n} R_k(x)\right| = 1$. From $x \in \bigcap_{k=1}^{n} I_k(x) \subset \bigcap_{k=1}^{n} R_k(x)$ and $\left|\bigcap_{k=1}^{n} R_k(x)\right| = 1$ we have $\{x\} = \bigcap_{k=1}^{n} R_k(x)$, therefore from Theorem 2 we obtain $x \in O_w$.

(b) Let $x \in O_s$ and let us assume $\bigcap_{k=1}^{n} I_k(x) \neq \bigcap_{k=1}^{n} R_k(x)$. From $\bigcap_{k=1}^{n} I_k(x) \subset \bigcap_{k=1}^{n} R_k(x)$ and $\bigcap_{k=1}^{n} I_k(x) \neq \bigcap_{k=1}^{n} R_k(x)$ it follows that there exists $y \in \bigcap_{k=1}^{n} R_k(x)$ such that $y \notin \bigcap_{k=1}^{n} I_k(x)$. We obtain there exists $i_m \in I$ such that $yP^m x$. As a result we obtain $yR^k x$ for all $i_k \in I$ and $yP^m x$. This contradicts to $x \in O_s$. Finally, we obtain $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$.

Conversely, let $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ and let us assume $x \notin O_s$. Hence, we have there exists $y \in A$ such that $y \in \bigcap_{k=1}^{n} R_k(x)$ and $y \in P_m(x)$ for some $i_m \in I$. From $y \in P_m(x)$ it follows $y \notin \bigcap_{k=1}^{n} I_k(x)$. This contradicts to $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ therefore $x \in O_s$.

(c) Let $x \in O_f$. If $y \in \bigcap_{k=1}^{n} R_k(x)$, then from Theorem 6.c it follows $y \in \bigcup_{k=1}^{n} I_k(x)$.

Conversely, let $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$ and let us assume $x \notin O_f$. Following we have there exists $y \in A$ such that $y \in \bigcap_{k=1}^{n} P_k(x)$. From $\bigcap_{k=1}^{n} P_k(x) \subset \bigcap_{k=1}^{n} R_k(x)$ and $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$ we obtain $y \in \bigcup_{k=1}^{n} I_k(x)$. Thus, we have there exists $i_m \in I$ such that $yI^m x$. As a result we obtain $y \notin P_m(x)$, i.e. $y \notin \bigcap_{k=1}^{n} P_k(x)$. This contradicts to $y \in \bigcap_{k=1}^{n} P_k(x)$.

Corollary 1. *For $x \in A$ the following statements hold:*

(a) $x \in O_s \backslash O_w$ if and only if $\{x\} \neq \bigcap_{k=1}^{n} R_k(x)$ and $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$;

(b) $x \in O_s \backslash O_w$ if and only if $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ and $\left|\bigcap_{k=1}^{n} R_k(x)\right| > 1$;

(c) $x \in A \backslash O_s$ if and only if $\bigcap_{k=1}^{n} R_k(x) \neq \bigcap_{k=1}^{n} I_k(x)$;

(d) $x \in O_s$ if and only if $\{x\} = \bigcap_{k=1}^{n} R_k(x)$ or ($\{x\} \neq \bigcap_{k=1}^{n} R_k(x)$ and $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$);

(e) $x \in O_f \backslash O_s$ if and only if $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$ and $\bigcap_{k=1}^{n} R_k(x) \neq \bigcap_{k=1}^{n} I_k(x)$.

Proof. From Theorem 9 it follows the proof of Corollary 1.

Corollary 2. *(a) If $x \in O_s$ and $\{x\} = \bigcap_{k=1}^{n} I_k(x)$, then $x \in O_w$;*

(b) If $x \in O_f$ and $\bigcap_{k=1}^{n} I_k(x) = \bigcup_{k=1}^{n} I_k(x)$, then $x \in O_s$.

Proof. (a) From Theorem 9 we have $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$. From $\{x\} = \bigcap_{k=1}^{n} I_k(x)$ it follows $\{x\} = \bigcap_{k=1}^{n} R_k(x)$ therefore from Theorem 2 we obtain $x \in O_w$.

(b) From Theorem 9 we have $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$. From $\bigcap_{k=1}^{n} I_k(x) \subset \bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$ and $\bigcap_{k=1}^{n} I_k(x) = \bigcup_{k=1}^{n} I_k(x)$ it follows $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$ therefore from Theorem 9 we obtain $x \in O_s$.

REFERENCES

[1] A. Alkan, G. Demange, D. Gale. Fair allocation of indivisible goods and criteria of justice. *Econometrica*, 59 (1991), 1023–1039.

[2] A. Chiang. Fundamental Methods of Mathematical Economics. MrGraw-Hill, New York. 1984.

[3] M. Inrtiligator. Mathematical Optimization and Economic Theory. Prentice-Hall, New York, 1971.

[4] Z. Slavov. Pareto Optimum in Mathematical Economy. Academic Open Internet Journal 3, 2000.

[5] K. Suzumura. Pareto principles from Inch to Ell. *Economies Letters* 70 (2001), 95–98.

Zdravko Dimitrov Slavov
Varna Free University
Department of Mathematics
e-mail: `blhrdezd@revolta.com`

ОПТИМАЛНИТЕ АЛТЕРНАТИВИ ПРИ ВЗЕМАНЕТО НА РЕШЕНИЕ

Здравко Димитров Славов

В настоящата работа се изучава концепцията за оптималност по Парето при вземането на решение в общество с крайно много индивиди. Разглеждат се отношенията на предпочитания на индивидите и три версии на оптимални алтернативи според Парето – слаба, силна и пълна.