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ON THE SUBCRITICAL AGE-DEPENDENT BRANCHING PROCESSES WITH TWO TYPES OF IMMIGRATION*

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The purpose of this paper is to present a probabilistic proof under week conditions of the convergence in probability of the subcritical age-dependent branching processes allowing two different types of immigration, i.e. one type in the state zero and another one according to the i.i.d. times of an independent ergodic renewal process.

Kew words: age-dependent branching processes; state-dependent immigration; convergence in probability

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1. Introduction. Let us consider the following population process $\{X(t)\}_{t\geq 0}$. At the random times τ_k, $k = 1, 2, \ldots$, a random number of individuals enters the population. An individual appearing at time τ_k becomes an ancestor of Bellman-Harris branching process with immigration in the state zero (BHBPIO) $\{Z(t)\}_{t\geq 0}$. The process $X(t)$ counts the number of individuals alive at time t and we call this model, Bellman-Harris branching process with immigration at zero state and an immigration of renewal type (BHBPIOR).

The intervals between successive immigration $T_1 = \tau_1, T_2 = \tau_2 - \tau_1, \ldots$ and the number of immigrants $\nu_1, \nu_2, \ldots,$ are assumed to be mutually independent random variables (i.r.v.). The r.v. T_k have common distribution function (d.f.) $G_0(t)$ and the r.v. ν_k are defined by common probability generating function (p.g.f.) $f_0(s)$. The BHBPIO $\{Z(t)\}_{t\geq 0}$ is governed by a lifetime distribution $G(t)$, an offspring p.g.f. $h(s)$, a p.g.f. $f(s)$ of the random number Y_i of immigrants in the state zero and the d.f. $K(t)$ of the duration X_i of staying in the state zero. It is assumed that $\int_0^\infty t dK(t) < \infty$.

We will use the definition of BHBPIO given by Mitov and Yanev (1985):

$$Z(t) = Z_{N(t)+1}(t - \xi(t))\mathbb{I}_{\{\xi(t)<t\}}, \xi(t) = S_{N(t)} + X_{N(t)+1}, Z(0) = 0, \tag{1.1}$$

where $\{Z_i(t)\}$ are independent Bellman-Harris branching processes starting with random number Y_i of particles, $N(t) = \max\{n \geq 0 : S_n \leq t\}$, $S_0 = 0$, $S_n = \sum_{i=1}^n U_i$, $U_i = X_i + \sigma_i$, $\sigma_i = \inf\{\ t : Z_i(t) = 0\}$ and $\mathbb{I}_{\{\cdot\}}$ is the indicator function.

Let us mention that the process defined by (1.1) could be interpreted as follows: starting from the zero state, the process stays at that state random time X_i with d.f.

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$K(t)$ and after that a random number Y_i of immigrants enters the population, according to the p.g.f. $f(s)$. The further evolution of each particle is independent and in accordance with a d.f. $G(t)$ of the life-time and the p.g.f. $h(s)$ of the offspring. Then the process hits zero after a random period σ_i, depending of the evolution of the inner BHBP $Z_i(t)$. The following evolution of the process could be presented as the replication of such i.i.d. cycles.

We introduce the following notations for the p.g.f. of the local characteristics of the processes

$$f(s) = \mathbb{E}s^{Y_1} = \sum_{k=1}^{\infty} f_k s^k, \quad h(s) = \sum_{k=0}^{\infty} p_k s^k, \quad f_0(s) = \mathbb{E}s^{\nu_1} = \sum_{k=0}^{\infty} q_k s^k.$$

It will be assumed:

$$(1.2) \qquad 0 < A = h'(1) < \infty, \quad m = f'(1) < \infty, \quad m_0 = f_0'(1) < \infty,$$

$$(1.3) \qquad G(t),\ G_0(t) \text{ and } K(t) \text{ are non-lattice},$$

$$(1.4) \qquad 0 < B = h''(1) < \infty, \quad n = f''(1) < \infty, \quad f_0''(1) = b_2 < \infty,$$

$$(1.5) \qquad r = \int_0^{\infty} x dG(x) < \infty, \quad \tilde{a} = \int_0^{\infty} x dK(x) < \infty, \quad r_0 = \int_0^{\infty} x dG_0(x) < \infty.$$

Note that $L(t) = \mathbb{P}\{X_i + \sigma_i \le t\}$ is non-lattice with $L(0) = 0$ and denote $\mu = \int_0^{\infty} t dL(t)$.

Let us mention that in the critical case for the first time the BHBPIOR was studied by Weiner (1991). Later on, Slavtchova-Bojkova and Yanev (1994) analyzed the model $X(t)$ with two types of immigration in the non-critical cases and the problem of determining necessary and sufficient conditions for the existence of a limiting distribution were investigated. The results in the subcritical case are proved under the strong assumption of an existence of higher ($n > 2$) moments of the individual characteristics.

The aim of this work is to prove the convergence in probability of the subcritical BHBPIOR $X(t)$ only under assumption that the first and second moments are finite. The main result is the following theorem.

2. Main result.

Theorem 2.1. *Let us assume that* (1.2) – (1.5) *hold. If* $A < 1$, *then*

$$\frac{X(t)}{t} \xrightarrow{\mathbb{P}} c,$$

($\xrightarrow{\mathbb{P}}$ means convergence in probability) as $t \to \infty$, *where* $c = m m_0 r/(1 - A)\mu r_0$.

Proof. To start the proof, at first we give an equivalent representation of the process $X(t)$.

Let $\{Z_{ij}(t)_{t\ge 0}\}$ be a doubly infinite collection of independent random processes each having the same distribution as the BHBPIO $\{Z(t)\}_{t\ge 0}$. Furthermore, let all these processes be assumed to be independent of the sets of r.v. $\{\tau_i\}$ and $\{\nu_i\}$. By going to the product space, we can assume that all the above mentioned random quantities are defined on a common probability space.

Define the renewal function $n(.)$ by setting $n(t) = k$ if $\tau_k \le t < \tau_{k+1}$, $k \ge 0$, $\tau_0 = 0$.

It now follows from the assumptions in section 1, that for each $t > 0$

(2.1) $$X(t) = \sum_{i=0}^{n(t)} \sum_{j=1}^{\nu_i} Z_{ij}(t - \tau_i) \quad a.s.$$

It is known for the subcritical BHBPIO $Z(t)$ (see Slavtchova and Yanev (1991)), that

$$\lim_{t\to\infty} \mathbb{E}Z(t) = mr/(1-A)\mu, \quad \mu = \int_0^\infty t dL(t),$$

and there exists stationary limit distribution, i. e.

$$\lim_{t\to\infty} \mathbb{P}\{Z(t) = k\} = \Phi_k = \mathbb{P}\{Z(\infty) = k\}, \quad \sum_{k=0}^{\infty} \Phi_k = 1, \quad \Phi(s) = \sum_{k=0}^{\infty} \Phi_k s^k, |s| \le 1$$

and

(2.2) $$\mathbb{E}Z(\infty) = \Phi'(1) = mr/(1-A)\mu \equiv a.$$

Let us denote

$$\begin{aligned} m_{ij}(t) &= \mathbb{E}Z_{ij}(t), \\ S(t) &= \frac{1}{t} \sum_{i=0}^{n(t)} \sum_{j=1}^{\nu_i} Z_{ij}(t - \tau_i), \\ S^\star(t) &= \frac{1}{t} \sum_{i=0}^{n(t)} \sum_{j=1}^{\nu_i} m_{ij}(t - \tau_i). \end{aligned}$$

To prove the theorem one need to check that for every $\varepsilon > 0$

$$\lim_{t\to\infty} \mathbb{P}\left\{ \left| \frac{X(t)}{t} - c \right| > \varepsilon \right\} = 0.$$

We have the following estimation:

$$\begin{aligned} \mathbb{P}\{|S(t) - c| > \varepsilon\} &\le \mathbb{P}\left\{|S(t) - S^\star(t)| > \frac{\varepsilon}{2}\right\} + \mathbb{P}\left\{|S^\star(t) - c| > \frac{\varepsilon}{2}\right\} \\ &= I_1 + I_2 \end{aligned}$$

Applying the Chebishev's inequality for I_1 we obtain

(2.3) $$I_1 \le \frac{\mathrm{Var}[S(t) - S^\star(t)]}{\varepsilon^2}.$$

Denote $F_t = \sigma(\tau_i, \nu_i, i = 1, 2, \ldots, n(t); n(t))$. Using that

$$\mathbb{E}[S(t) - S^\star(t)] = \mathbb{E}\{\mathbb{E}[S(t) - S^\star(t)|F_t]\} = 0,$$

f or the variance we have

(2.4) $$\begin{aligned} \mathrm{Var}[S(t) - S^\star(t)] &= \mathbb{E}[S(t) - S^\star(t)]^2 \\ &= \mathbb{E}\{\mathbb{E}[S(t) - S^\star(t)]^2 | F_t\} \\ &= \frac{1}{t^2} \mathbb{E}\left[\sum_{i=0}^{n(t)} \sum_{j=1}^{\nu_i} \mathrm{Var} Z_{ij}(t - \tau_j)\right]. \end{aligned}$$

Set $d = \sup_t \mathbb{E}\left(Z_{ij}(t) - m_{ij}(t)\right)^2$. Using Wald's inequality and the fact that $|\mathrm{Var} Z_{ij}(t-\tau_i)| < d < \infty$, as $t \to \infty$, from (2.3) and (2.4) we obtain

$$\mathrm{Var}[S(t) - S^\star(t)] = \mathbb{E}[S(t) - S^\star(t)]^2$$

$$= \mathbb{E}\left(\mathbb{E}\left((S(t) - S^\star(t))^2 \,|F_t\right)\right)$$

$$= \frac{1}{t^2}\mathbb{E}\sum_{i=0}^{n(t)}\sum_{j=1}^{\nu_i}\mathbb{E}\left((Z_{ij}(t-\tau_i) - m_{ij}(t-\tau_i))^2 \,|F_t\right)$$

$$\le \frac{d}{t^2}\mathbb{E}\sum_{i=0}^{n(t)}\nu_i = \frac{d}{t^2}\mathbb{E}n(t)\mathbb{E}\nu_1 \le \frac{2d\mathbb{E}\nu_1}{t\mathbb{E}\tau_1}$$

for all sufficiently large t, since

$$n(t)/t \xrightarrow{\mathbb{P}} 1/\mathbb{E}\tau_1$$

and

$$\mathbb{E}n(t)/t \to 1/\mathbb{E}\tau_1, \text{ as } t \to \infty.$$

Thus,

$$I_1 \xrightarrow{\mathbb{P}} 0, \text{ as } t \to \infty. \tag{2.5}$$

Now, note only that $c = a\mathbb{E}\nu_1/\mathbb{E}\tau_1$, where a is defined by (2.2). One has

$$S^\star(t) - c = \frac{1}{t}\sum_{i=0}^{n(t)}\sum_{j=1}^{\nu_i} m_{ij}(t-\tau_i) - c$$

$$= \frac{1}{t}\sum_{n(t-T)}^{n(t)}\sum_{j=1}^{\nu_i} m_{ij}(t-\tau_i) + \frac{1}{t}\sum_{i=0}^{n(t-T)}\sum_{j=1}^{\nu_i}(m_{ij}(t-\tau_i) - a) +$$

$$+a\left(\frac{1}{t}\sum_{i=0}^{n(t-T)}\nu_i - \frac{\mathbb{E}\nu_1}{\mathbb{E}\tau_1}\right) \equiv I_3 + I_4 + I_5.$$

Now

$$0 \le I_3 \le \frac{R_1}{t}\sum_{n(t-T)}^{n(t)}\nu_i, \quad a.s.$$

where $R_1 = \sup_t m_{ij}(t) < \infty$.

Let $T = o(t) \to \infty$, $t \to \infty$. Therefore,

$$\mathbb{P}\{I_3 > \varepsilon\} \le \frac{R_1}{\varepsilon t}\mathbb{E}\sum_{n(t-T)}^{n(t)}\nu_i = \frac{R_1}{\varepsilon t}\mathbb{E}\nu_1\mathbb{E}\left(n(t) - n(t-T)\right) = o(1), \tag{2.6}$$

as $t \to \infty$.

Denoting

$$r(T) = \sup_{x \ge T}|m_{ij}(x) - a|,$$

we see that

$$|I_4| \leq \frac{r(T)}{t} \sum_{i=0}^{n(t-T)} \nu_i$$

and, therefore, for any fixed $\varepsilon > 0$

$$\mathbb{P}\{|I_4| > \varepsilon\} \leq \frac{r(T)}{t}\,\mathbb{E}\sum_{i=0}^{n(t-T)} \nu_i = \frac{r(T)\mathbb{E}\nu_1}{t}\mathbb{E}n(t-T) \leq R_2 r(T) \to 0 \tag{2.7}$$

as first $t \to \infty$ and then $T \to \infty$ and $R_2 = \mathbb{E}\nu_1/\mathbb{E}\tau_1$. Finally, by the law of the large numbers and the renewal theorem (see Feller (1971), Section XI.6)

$$\frac{1}{t}\sum_{i=0}^{n(t-T)} \nu_i = \frac{n(t-T)}{t}\frac{1}{n(t-T)}\sum_{i=0}^{n(t-T)} \nu_i \to \frac{\mathbb{E}\nu_1}{\mathbb{E}\tau_1} \text{ a.s.}$$

and, therefore,

$$I_5 \xrightarrow{\mathbb{P}} 0 \tag{2.8}$$

as $t \to \infty$. From (2.5)-(2.8) the desired statement follows.

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ВЪРХУ ДОКРИТИЧНИ РАЗКЛОНЯВАЩИ СЕ ПРОЦЕСИ, ЗАВИСЕЩИ ОТ ВЪЗРАСТТА С ДВА ТИПА ИМИГРАЦИЯ

Марусия Н. Божкова

Разгледани са докритични разклоняващи се процеси на Белман-Харис с два типа имиграция. Целта на работата е получаване на сходимост по вероятност на процесите само при условие за съществуване на крайни първи и втори факториални моменти на индивидуалните характеристики на процесите.