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MODIFICATIONS OF THE CONCEPT OF PERFECT NUMBER*

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Some modifications (extensions) of the concept of a perfect number are introduced. Some of their properties and examples are discussed.

1. Introduction. Perfect numbers and the problems related to them are among the most popular concepts in number theory. Here we shall formulate some modifications of these numbers and shall discuss some of their properties, giving a series of open problems, related to them.

It is well known that for the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, α_1, $\alpha_2, \dots$ $\alpha_k \geq 1$ are natural numbers and $p_1 < p_2 < \cdots < p_k$ are different prime numbers, the Jordan's (φ_s) (s – a natural number), Euler's (φ) and σ functions are defined by:

$$\varphi_s(n) = \prod_{i=1}^{k} p_i^{s(\alpha_i - 1)}(p_i^s - 1),$$

$$\varphi(n) = \varphi_1(n),$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

Let $P_d(n)$ and $\tau(n)$ be, respectively, the product and the number of all divisors of n. It is well know, that for every natural number n:

(1) $$P_d(n) = \sqrt{n^{\tau(n)}}.$$

It is noted in the Jelenski's book [1] that some of the old Greek mathematicians called perfect numbers these numbers that are equal to the product of their proper divisors. For example, $6 = 1.2.3$, $8 = 1.2.4$, $10 = 1.2.5$, $14 = 1.2.7$, etc. These numbers can be called *multiplicative perfect numbers,* for instance of the additive perfect numbers, that are equal to the sum of their proper divisors. The following two definitions are more explicit:

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Definition 1. *The natural number n is an (ordinary) additive perfect number if and only if (iff)*

$$\sigma(n) \equiv \sum_{d|n} d = 2n.$$

Definition 2. *The natural number n is a multiplicative perfect number iff*

$$P_d(n) \equiv \prod_{d|n} d = n^2.$$

Lemma 1. *The necessary condition for a natural number $n > 1$ to be a multiplicative perfect number is $\tau(4) = 4$.*

Proof. Follows directly from (1) and Definition 2.

Corollary 1. *If $n > 1$ is a multiplicative perfect number, then n has exactly two different proper divisors larger than 1.*

Corollary 2. *Number $n > 1$ is a multiplicative perfect number, iff n is a cube of a prime number or n is a product of two different prime numbers.*

Corollary 3. *The multiplicative perfect numbers are infinitely many.*

We must note that for the additive perfect numbers no similar to the last result has been obtained up to now.

Definition 3. *Let g be a fixed arithmetic function. The natural number n is g-additive perfect number iff*

$$\prod_{d|n} g(d) = 2g(n). \tag{2}$$

Definition 4. *Let g be a fixed arithmetic function. The natural number n is g-multiplicative perfect number iff*

$$\prod_{d|n} g(d) = (g(n))^2.$$

Obviously, Definition 1 is a particular case of Definition 3 and Definition 2 is a particular case of Definition 4 for $g(n) = n$.

We must note that for each natural number n that is a g-multiplicative perfect number, if $g(n) > 0$, then for every positive real number $c \neq 1$ n is a $\log_c g(n)$-additive perfect number.

The following examples hold.

Example 1. Let $g(n) = c^n$ for some positive real number $c \neq 1$. Then the g-multiplicative perfect numbers coincide with the classical additive perfect numbers (from Definition 1).

Example 2. The natural numbers of the form p^2, where p is a prime number, are τ-additive perfect numbers. The validity of this fact follows from equality

$$\sum_{d|n^2} \tau(d) = \tau(n)\tau(n^2)$$

(see [2]).

Theorem 1. *There are no φ_2-, φ_3- and φ_4-additive perfect numbers.*

Proof. Let us assume that the natural number n is a φ_2-additive perfect number. Since

$$\sum_{d|n} \varphi_s(d) = n^s$$

for every natural number s, then for $s = 2$ the Diophantine equation

$$\prod_{i=1}^{k} p_i^{2\alpha_i} = 2\prod_{i=1}^{k} p^{2(\alpha_i-1)}(p_i^2 - 1)$$

i.e.,

$$\prod_{i=1}^{k} p_i^2 = 2\prod_{i=1}^{k}(p_i^2 - 1) \tag{3}$$

ought to have a solution. Therefore, $p_1 = 2$, because p_1 is the smallest prime divisor of n and n is divisible by 2. Hence, (3) has the form

$$2\prod_{i=2}^{k} p_i^2 = 3\prod_{i=2}^{k}(p_i^2 - 1). \tag{4}$$

All ($k - 1$ in number) multipliers $(p_i^2 - 1)$ $(2 \le i \le k)$ are even numbers, while in the left side we divide by 2 but not – by 4. Therefore, $k = 2$. Hence (4) has the form:

$$2p_2^2 = 3(p_2^2 - 1)$$

or

$$p_2^2 = 3$$

that is impossible. Therefore, Theorem 1 is valid for the case of φ_2-additive perfect number. For the two other cases the proof is similar.

Interesting is the following

Open problem 1. Is there a natural number $s > 1$ for which there exists a φ_s-additive perfect number?

Theorem 2. *The natural number n is φ-additive perfect number iff $n = 2^k$, for each natural number k.*

Proof. Let $n = 2^k$. Then $2\varphi(n) = 2\varphi(2^k) = 2.2^{k-1} = 2^k = n = \sum_{d|n} \varphi(d)$. Therefore, from (2) for $g(n) = \varphi(n)$, we obtain that n is a φ-additive perfect number.

Let us assume that the natural number n is a φ-additive perfect one. Therefore,

$$\sum_{d|n} \varphi(d) = 2\varphi(n).$$

But

$$\sum_{d|n} \varphi(d) = n.$$

Therefore,

$$n = 2\varphi(n). \tag{5}$$

Hence, n is an even number. Let $n = 2^k m$, where $m \ge 1$ is an odd number. From (5) and from the fact that φ is a multiplicative function, it follows that

$$n = 2^k m = 2\varphi(n) = 2\varphi(2^k m) = 2\varphi(2^k)\varphi(m) = 2^k\varphi(m),$$

i.e.,

$$m = \varphi(m),$$

that is possible only if $m = 1$. Therefore, $n = 2^k$ and Theorem 2 is proved.

From Theorem 2 it follows the following

Corollary 4. *There are infinite many φ-additive perfect numbers.*

Corollary 5. *There are no odd φ-additive perfect numbers.*

Of course, we can note (for more details see, e.g., [3]) that up to now it is not clear whether the classical additive perfect numbers are infinitely many. Also, we do not know whether there is at least one odd number $n \geq 1$ that is an additive perfect number.

Euler proved (see [4]) that all even additive perfect numbers have the form:

$$n = 2^{p-1}(2^p - 1),$$

where $2^p - 1$ is a prime number.

The following open problems are interesting:

Open problem 2. To find all arithmetic (in particular, at least all multiplicative) functions g for which there exists no g-additive perfect number.

As we see above, φ_2, φ_3 and φ_4 are such functions.

Open problem 3. For each arithmetic (in particular, multiplicative) function g to find g-additive perfect numbers, if they exist. Is there an infinite set of numbers that are not g-additive perfect ones?

Open problem 4. To find a procedure for obtaining of all g-multiplicative perfect numbers for separate classical arithmetic functions.

Finally, we can introduce the following

Definition 5. *Let g be a fixed arithmetic function and p be a fixed prime number. The natural number n is a (g, p)-power perfect number iff*

$$\prod_{d|n} g(d) = p^{g(n)}.$$

Theorem 3. *$n = 4$ is the unique $(2^{\varphi(n)}, 2)$-power perfect number.*

The proof is similar to the above one.

Open problem 5. To find a procedure for obtaining of all (g, p)-power perfect numbers for separate classical arithmetic functions and for different prime numbers p.

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