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ALGEBRAIC MODELS FOR TWO-PLAYER GAMES WITH ITERATIONS

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We study algebraic structures modelling perfect information games with two players, with game constructing operations: choices of first and second player, composition, dualization, weak and strong iteration. We axiomatize the valid identities in such structures thus giving a structural-algebraic characterization of equivalence between games.

1. Introduction. Parikh initiated in [1] a formal logical analysis of Game theory, by introducing a formal system, called Game Logic, for modelling of, and reasoning about, two-player games. This approach has been advanced in [3], where certain *game algebras* over effectivity functions for the players, naturally arising from the semantics of Game Logic, were introduced. These game algebras involve the following *basic game operations:* choices of first and second player, composition and dualization. We have established in [3] a complete axiomatization of the valid identities of these game algebras.

Here we extend the basic game algebras with *game iterations,* and axiomatize the valid identities in the resulting structures.

2. Game models and effectivity functions. We will deal with *two-player games,* with Player 1 and Player 2, where Player 1 makes the first move in the game. Here we adopt a *structural* approach to games, whereby they are considered as built from *atomic* games by applying certain natural *game operations.*

The game models we consider are based on abstract **'game boards',** consisting of a **set of game states**, where a play of a game effects a transition from a state (*initial state*) to a state (*outcome state*), according to the choices of moves/strategies of each of the individual players. For every game G each player has certain *powers* to determine the outcome, which can be modelled by sets of states for which the player is *effective.*

Definition 1. *Given a set of game states S and a game G on S, a* ***player P is effective for a set of states $X \subseteq S$ in the game G with an initial state*** *$s \in S$ if P has a move/strategy in G which can guarantee that, if the game begins at the state s and it ends, the outcome state will be in X, irrespective of the moves/strategies of the other player.*

Thus, with each set of states S, game G on S, and a player i, for i =1,2, we associate an **effectivity function** $\rho^i_G : S \to \mathbf{P}(\mathbf{P}(\mathbf{S}))$, defined as follows:

$$X \in \rho^i_G(s) \text{ iff Player i is effective for } X \text{ in } G \text{ with an initial state } s.$$

The resulting structure will be called a **game board.**

The effectivity functions associated with atomic games are given *a priory*, and they must some natural *effectivity conditions*: for any atomic game a, $s \in S$, and $X \subseteq S$:

1. **Consistency of powers (CON):**

   if $s\rho^1_a X$ then not $s\rho^2_a(S - X)$, and likewise with 1 and 2 swapped.

2. **Upwards monotonicity (MON)***:*

   if $s\rho^i_a X$ and $X \subseteq Y \subseteq S$, then $s\rho^i_a Y$.

2.1. Basic game operations. The basic set of game operations which we consider here consist of:

- **choice of** Player 1, denoted by $\vee$: given two games g_1 and g_2, the game $g_1 \vee g_2$ begins with Player 1 choosing which of the games g_1 and g_2 should be played further, followed by a play of that chosen game.
- **choice of** Player 2, denoted by $\wedge$: the game $g_1 \wedge g_2$ is defined just like $g_1 \vee g_2$, except that now the initial choice is made by Player 2.
- **composition,** denoted by $\circ$: given two games g_1 and g_2, the game $g_1 \circ g_2$ begins with playing g_1, followed by playing g_2 with initial state being the outcome of the play of g_1.
- **dualization (swap of roles):** the **dual** of a game g is the game g^d obtained from g, by swapping the roles of the two players.

Note that $\wedge$ is definable in terms of $\vee$ and ^{d} by De Morgan's law:

$$g_1 \wedge g_2 = (g_1^d \vee g_2^d)^d.$$

In addition, mainly for technical convenience, we assume that one of the atomic games is the '*idle game*' ι, where no transition occurs. The effectivity functions ρ^i_ι of the idle game have the obvious definition: $s\rho^i_\iota X$ iff $s \in X$.

2.2. The basic algebra of games. Applying the basic game operations to atomic games we can build complex games. That process can be formalized in a standard algebraic fashion, by using terms in a suitable formal language.

Definition 2. *The* ***basic game language*** *GL consists of:*

- *a set of* ***atomic games*** $\mathcal{G}_{at} = \{g_a\}_{a \in A}$, *where* $\iota = g_0$.
- ***game operations:*** $\vee, ^d, \circ$.

Definition 3. ***Game terms:***

- *Every atomic game is a game term.*
- *If G, H are game terms then G^d, $G \vee H$ and $G \circ H$ are game terms.*

Now, we define $G \wedge H := (G^d \vee H^d)^d$ and $G \leq H$ iff $G \vee H = H$.

Atomic games and their duals will be called **literals.** Compositions of idle literals (ι or ι^d) will be called **idle game terms.**

Given a game board B, the atomic effectivity functions can now be extended to effectivity functions $\{\rho^i_G\}_{G \in \mathcal{G}; i=1,2}$ for *all game terms*, following the recursive definitions, first introduced in [1]:

- $s\rho^1_{G^d} X$ iff $s\rho^2_G X$;
- $s\rho^2_{G^d} X$ iff $s\rho^1_G X$;
- $s\rho^1_{G_1 \vee G_2} X$ iff $s\rho^1_{G_1} X$ or $s\rho^1_{G_2} X$;
- $s\rho^2_{G_1 \vee G_2} X$ iff $s\rho^2_{G_1} X$ and $s\rho^2_{G_2} X$;
- $s\rho^1_{G_1 \circ G_2} X$ iff there exists Z such that $s\rho^1_{G_1} Z$ and $z\rho^1_{G_2} X$ for each $z \in Z$;
- $s\rho^2_{G_1 \circ G_2} X$ iff there exists Z such that $s\rho^2_{G_1} Z$ and $z\rho^2_{G_2} X$ for each $z \in Z$.

Proposition 4. *Each effectivity conditions propagates over all effectivity functions.*

Definition 5. *Two game terms G_1 and G_2 are **equivalent,** denoted $G_1 \sim G_2$ if on every game board they are assigned the same effectivity functions. If $G_1 \sim G_2$ we say that $G_1 = G_2$ is a **valid game identity**.*

The valid game identities give a structural-algebraic characterization of equivalence between games. In [3] we provide a complete axiomatization **GA** of the valid identities of the basic game algebra.

2.3. Adding game iterations. Here we extend the basic game algebra by adding *game iterations*. Intuitively, iteration of a game means finite or infinite consecutive repetition of that game. Depending on which player decides when the iteration ends, and when that decision is taken, two basic types of iteration emerge:

1. **Weak iteration** $g^{(*)}$: given a game g, the game $g^{(*)}$ consists in n consecutive rounds of playing g, for some number $n \geq 0$ determined by Player 1 *at the beginning of the game.*
2. **Strong iteration** $g^{[*]}$: given a game g, the game $g^{[*]}$ consists in n consecutive rounds of playing g, for some number $n \geq 0$ determined by Player 1 *at the end of the last round*.

By double dualization: $((g^d)^{(*)})^d$ and $((g^d)^{[*]})^d$ we can define iterations where Player 2 is in control.

For any $n \geq 0$, the *n***-repetition of a game** g is defined recursively by $g^0 = \iota$; $g^{n+1} = g^n \circ g$.

If we are allowed the liberty of using *infinite* game terms, then the weak and strong iterations can be explicitly defined as follows:

$$g^{(*)} = \iota \vee g \vee g^2 \vee \cdots = \bigvee_{n=0}^{\infty} g^n;$$

$$g^{[*]} = \iota \vee g \circ (\iota \vee g \circ (\ldots) \ldots).$$

Instead, we will define these iterations implicitly by means of equations, using the fact that their effectivity functions can be determined as least fixed points of certain *effectivity functors* (operators on effectivity functions).

Proposition 6. *For any finite set of states* $\mathbf{S}$ *and a game* g *on* $\mathbf{S}$:

1. $\rho^1_{g^{(*)}}$ *is the least fixed point of the monotone operator* $\mathfrak{F}^1_w$ *defined by* $\mathfrak{F}^1_w(\mathbf{F}) = \mathbf{I} \cup \mathbf{F} \circ \rho^1_g$.
2. $\rho^2_{g^{(*)}}$ *is the greatest fixed point of the monotone operator* $\mathfrak{F}^2_w$ *defined by* $\mathfrak{F}_w(\mathbf{F}) = \mathbf{I} \cap \mathbf{F} \circ \rho^2_g$.
3. $\rho^1_{g^{[*]}}$ *is the least fixed point of the monotone operator* $\mathfrak{F}^1_s$ *defined by* $\mathfrak{F}^1_s(\mathbf{F}) = \mathbf{I} \cup \rho^1_g \circ \mathbf{F}$
4. $\rho^2_{g^{[*]}}$ *is the greatest fixed point of the monotone operator* $\mathfrak{F}^2_s$ *defined by* $\mathfrak{F}_s(\mathbf{F}) = \mathbf{I} \cap \rho^2_g \circ \mathbf{F}$.

where $\mathbf{I} = \rho^1_\iota = \rho^2_\iota$.

Theorem 7. *The following identities and quasi-identities, added to the system* ***GA,*** *render a complete axiomatization of the identities for the game algebra with iterations.*

1. $g^{(*)} = \iota \vee g^{(*)} \circ g$;
2. $g^{[*]} = \iota \vee g \circ g^{[*]}$;
3. $g \circ h \leq h \Rightarrow g^{(*)} \circ h \leq h$;
4. $h \circ g \leq h \Rightarrow h \circ g^{[*]} \leq h$.

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АЛГЕБРИЧНИ МОДЕЛИ ЗА ИГРИ ЗА ДВАМА ИГРАЧИ С ИТЕРАЦИИ

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В тази статия изследваме алгебрични структури моделиращи игри с перфектна информация за двама играчи, със следните операции върху игри: избори за първия и за втория играч, композиция, дуализация, слаба и силна итерация. Тук ние аксиоматизираме алгебричните тъждества в тези структури, с което получаваме структурно-алгебрична характеризация на еквивалентност между игри от разглеждания клас.