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ON THE STABILIZING FEEDBACK CONTROLS*

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It has been proved by R. Brockett that, contrary to the case of linear systems, there exist small-time locally controllable nonlinear systems that can not be stabilized by means of stationary continuous feedback law. Following an idea of H. Hermes, we propose an approach for constructing discontinuous stabilizing feedback controls for a class of nonlinear systems defined in a neighbourhood of a closed set.

1. Introduction. Let S be a closed subset of R^n and let us consider the following control system:

(1) $$\dot{x}(t) \;=\; f(x(t), u(t)),$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, and $f : R^n \times R^n \to R^n$ is a smooth map. Starting from states close to the set S we want to steer S and to stay always close to S. To explain the difficulties related to this problem, let us consider the case when the set S is a point: Let $S = \{0\}$ and let us assume that the control system (1) is small-time local controllable at the origin, i.e. for any $T > 0$ there exists a neighbourhood Ω of the origin such that for any point $x_0 \in \Omega$ there exists an open-loop control $u_{x_0}(.)$ that steers the point x_0 to the origin in time not greater than T. Since open-loop controls are very sensitive to disturbances, they can lead to very bad practical results (cf., e.g., [16, Chap. 1, §4]). Taking this into account, one can try to find a feedback control law that asymptotically stabilizes the system. Such a feedback has the advantage of compensating automatically all random perturbations (when they are sufficiently small). A classical result (cf., e.g., [16, Thm. 7, p.134]) shows that the small-time local controllability of a linear control system implies that this system can be asymptotically stabilized by means of a stationary continuous feedback. A similar result does not hold true even for analytic nonlinear control systems. For example, it is pointed in [2] that the following three-dimensional control system

(2) $$\dot{x} = u,\ \dot{y} = v,\ \dot{z} = vx - uy$$

is small-time locally (and even globally) controllable at the origin but does not satisfy the Brockett necessary condition (cf. [2]), and hence can not be asymptotically stabilized by means of a continuous feedback law. To overcome the problem of impossibility to stabilize many controllable nonlinear systems by means of continuous feedback control laws, two main approaches have been proposed:

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(i) asymptotic stabilization by means of continuous time-varying feedback laws (cf. for example [7], [8]);

(ii) asymptotic stabilization by means of discontinuous feedback laws (cf. for example [11], [6]).

In this paper, we extend the results of [11] by studying the problem of small-time local asymptotic stabilizability (STLAS) with respect to a closed set using a more general class of high order control variations. The paper is organized as follows: A class of high order control variations with respect to a closed set is defined in section 2 and a sufficient condition for the STLAS property is proved. The proof of this sufficient condition can be used for constructing explicitly stabilizing feedback controls. To show the effectiveness of the proposed approach, a class of variations of second order are used to stabilize the control system (2) in a neighbourhood of the origin. Some simulation results (obtained by the system MAPLE V) are also presented at the end of the paper.

2. The main result. Throughout the paper, we shall use some notations and definitions we introduce in this section: Let S be an arbitrary closed subset of R^n. Let $\delta > 0$ and S_δ be the closed neighbourhood of the set S consisting of all points x for which $\mathrm{dist}_S(x) \leq \delta$. Here $\mathrm{dist}_S\,(x)$ denotes the distance between the point x and the set S, i.e.

$$\mathrm{dist}_S(x) \;:=\; \inf\{\|x - s\| \;|s \in S\}.$$

Let y belong to the boundary ∂S of the set S. A vector $\xi \in R^n$ is called a proximal normal to S at y provided there exists a real number $r > 0$ so that the point $y + r\xi$ has closest point y in S. The set of all proximal normals at a point y is a cone, and it is denoted by $N^p_S(y)$ (for a detailed treatment of proximal analysis and its applications, see for example [4], [5]).

Let $U \subset R^m$ and $f : R^n \times R^n \to R^n$ be a smooth map. We consider the following control system:

$$\dot{x}(t) \;=\; f(x(t), u(t)), \tag{3}$$

where $x(t) \in R^n$ is the state and $u(t) \in U$ is the control.

Let $T > 0$. Any function $k : S_\delta \times [0, T) \to U$ is called a (nonstationary) feedback. Any countable, strictly increasing sequence $\pi = \{t_i\}_{i=0}^{\infty}$ with $t_0 = 0$, and such that $\lim\limits_{i\to\infty} t_i = T$ is called a partition of the interval $[0, T)$. The trajectory $z_\pi(x_0, k, t),\ t \in [0, T)$, associated to the feedback $k : S_\delta \times [0, T) \to U$ and to any given partition π of $[0, T)$ is defined in the following step-by-step manner: Let x_0 be an arbitrary point from S_δ. Then between t_0 and t_1, $z_\pi(x_0, k, \cdot)$ is an absolutely continuous function (if such a function exists, of course) such that $z_\pi(x_0, k, 0) \;=\; x_0$ and for almost every t from $[t_0, t_1]$ the following relation holds true

$$\dot{z_\pi}(x_0, k, t) \;=\; f(z_\pi(x_0, k, t), k(x_0, t)).$$

We set $x_1 := z_\pi(x_0, k, t_1)$. Assuming that $x_1 \in S_\delta$, we restart the system with control $k(x_1, \cdot)$, i.e. between t_1 and t_2, $z_\pi(x_0, k, \cdot)$ is an absolutely continuous function such that $z_\pi(x_0, k, t_1) \;=\; x_1$ and for almost every t from $[t_0, t_1]$ the following relation holds true

$$\dot{z_\pi}(x_0, k, t) \;=\; f(z_\pi(x_0, k, t), k(x_1, t)).$$

We define $z_\pi(x_0, k, \cdot)$ on $[t_i, t_{i+1}], i = 2, 3, \ldots$ in the same way.

Definition 1. *It is said that the system (3) is small-time locally asymptotic stabilizable (STLAS) with respect to the set S iff for every $T > 0$ there exist a positive number δ such that for every point $x_0 \in S_\delta \setminus S$ there exist a partition π of some subinterval $[0, T_\pi) \subset [0, T]$ and a feedback $k : S_\delta \times [0, T_\pi) \to U$ such that the corresponding trajectory $z_\pi(x_0, k, \cdot)$ is well defined on $[0, T_\pi)$ and satisfies the following conditions:*
(a) $z_\pi(x_0, k, t) \in S_\delta$ for every $t \in [0, T_\pi)$;
(b) $\lim\limits_{t \to T_\pi} dist_S(z_\pi(x_0, k, t)) = 0$.

Our approach for studying the STLAS property is based on a suitably defined class of high-order control variations. To define it, we consider the following sets of functions: First, by $\mathcal{P}$ we denote the set of all functions $p(t), t \in R$, of the following type: $p(t) = \sum_{i=1}^{k} p_i t^{q_i}$, where $1 \leq q_1 < q_2 < \cdots < q_k$, and $0 < p_i, i = 1, \ldots, k$. Further, we shall also use the notations $o(t), o_1(t), \ldots$, to indicate any family of vector functions $o(t) : R^n \to R^n$, parameterized on t, continuous in (t, x) and such that for some $r > 0$ and $\omega > 1$ the ratio $o(t, x)/t^\omega$ is bounded uniformly with respect to $x \in S_r$. Let $\mathcal{A}_S^0$ be the set of all families of smooth vector functions $a(t) : R^n \to R^n$, parameterized on t, continuous in (t, x) and such that for every element $a(t)$ from $\mathcal{A}_S^0$ there exist some positive reals r, θ and M such that $\|a(t, x)\| \leq M.t^\theta.\mathrm{dist}_S(x)$ for all $x \in S_r$.

Definition 2. *Let S be an arbitrary closed subset of R^n and $A : R^n \to R^n$ be a smooth function. It is said that A is a control variation of order α iff there exist positive real numbers r, P and T, an element $p \in \mathcal{P}$, two families of vector functions $o(t)$ and $a(t) \in \mathcal{A}_S^0$ such that for every point x from $S_r \setminus S$ and each $t \in [0, T]$ there exists a piecewice continuous control function $u_t : [0, p(t)] \to U$ such that the corresponding trajectory $z(x, u_t, \cdot)$ is defined on $[0, p(t)]$, $|p(t)| \leq Pt$ and*

$$z(x, u_t, p(t)) = x + t^\alpha A(x) + a(t, x) + o(t^\alpha, x). \tag{4}$$

By $\mathcal{V}_S^\alpha$ we denote the set of all control variations of order α.

Remark 1. A natural question is to ask how to construct elements of the set $\mathcal{V}_S^\alpha$ when a control system is given in the form of differential equation or differential inclusion. Partial answers are given in [1], [9], [10], [11], [12], [13], [14], [15], [17], [18], [19] and etc.

Definition 3. *Let $r > 0$ and $T > 0$. It is said that $\mathcal{A}$ is a regular subset of the set $\mathcal{V}_S^\alpha$ on the set $S_r \times [0, T]$ provided that there exist positive constants $\omega > \alpha$, θ, L, M, N, C and P such that for every $A \in \mathcal{A}$ with corresponding $p(.)$, $a(t, x)$ and $o(t, x)$ (according to definition 1.), the following relations hold true:*
i) $\|A(x) - A(y)\| \leq L\|x - y\|$ for all x, y from S_r;
ii) $\|A(x)\| \leq C$ for all x from S_r;
iii) $\|a(x, t))\| \leq M.t^\theta.dist_S(x)$ for all x from S_r and all $t \in [0, T]$;
iv) $\|o(x, t^\alpha))\| \leq N.t^\omega$ for all x from S_r and all $t \in [0, T]$;
v) $|p(t)| \leq P.t$ for all $t \in [0, T]$;
vi) for all x from S_r and all $t \in [0, T]$

$$z(x, u_t, p(t)) = x + t^\alpha A(x) + a(t, x) + o(t^\alpha, x),$$

where u_t is the admissible control defined on $[0, p(t)]$ (according to Definition 1).

Now we can formulate the main result

Theorem 1. *Let S be a closed subset of R^n, $\mu > 0$, $T_0 > 0$, $\delta_0 > 0$ and let $\mathcal{A}$ be a regular subset of the set $\mathcal{V}_S^\alpha$ on $S_{\delta_0} \times [0, T_0]$. Let us assume that whenever y belongs to the boundary ∂S of the set S and $\xi \in N_S^p(y)$ there exists $A \in \mathcal{V}$ for which*

$$< \xi, A(y) > \leq -\mu . \|\xi\|. \tag{5}$$

Then the control system (3) *is STLAS with respect to S.*

3. Numerical results. The idea of the proof of the main result is to move towards the set S using suitable high-order control variations defined on suitable intervals of time. For an illustrative example, we consider the following three-dimensional control system

$$\begin{array}{lll} \dot{x}_1 = u, & x_1(0) = 0, & u \in [-1, 1], \\ \dot{x}_2 = v, & x_2(0) = 0, & v \in [-1, 1], \\ \dot{x}_3 = vx_1 - ux_2, & x_3(0) = 0, & \end{array}$$

which is small-time locally controllable at the origin but does not satisfy the Brockett necessary condition (cf. [2]), and hence can not be asymptotic stabilized by means of a continuous feedback law. Let $x = (x_1, x_2, x_3)^T$, $B(x) = (1, 0, -x_2)^T$ and $C(x) = (0, 1, x_1)^T$. Then the Lie bracket $[B, C](\cdot)$ of the vector fields $B(\cdot)$ and $C(\cdot)$ is

$$[B, C](x) := \partial C(x)B(x) - \partial B(x)C(x) = (0, 0, 2)^T,$$

where by $\partial B(\cdot)$ and $\partial C(\cdot)$ are denoted the Jacobians of $B(\cdot)$ and $C(\cdot)$, respectively.

The considered control system belongs to the class of the so called "symmetric" control systems (cf. [3]). So, it is natural to assume that all nonvanishing Lie brackets belong to the set of high-order control variations. Indeed, studying the set of admissible trajectories of this control system, one can find the following control variations of second order: Let $\beta = (\beta_1, \beta_2, \beta_3)^T$, where $\beta_i \in [-1, 1]$, $i = 1, 2, 3$. We define the following admissible control:

$$u_\beta(\tau) := \begin{cases} u = \beta_3, \ v = 0, & \text{for } \tau \in [0, t); \\ u = 0, \ v = 1, & \text{for } \tau \in [t, 2t) \\ u = -\beta_3, \ v = 0, & \text{for } \tau \in [2t, 3t); \\ u = 0, \ v = -1, & \text{for } \tau \in [3t, 4t) \\ u = \beta_1, \ v = \beta_2, & \text{for } \tau \in [4t, 4t + t^2); \end{cases}$$

It could be directly verified that the corresponding trajectory $z(x, u_\beta, \cdot)$ is defined on $[0, 4t + t^2]$ and

$$z(x, u_\beta, 4t + t^2) = x + t^2 \left(\beta_1 B(x) + \beta_2 C(x) + \beta_3 [B, C](x)\right).$$

Now we define the following stabilizing feedback $u_{\beta(x)}$ on $\Omega \times [0, 4t_x + t_x^2]$, where

$$\Omega := \left\{x = (x_1, x_2, x_3)^T : \ |x_1| \leq 1, \ |x_2| \leq 1, \ |x_3| \leq 2\right\} \text{ and } t_x := \sqrt{\|x\|}.$$

Since $B(0)$, $C(0)$ and $[B, C](0)$ are linearly independent, every point $x \in \Omega$ can be present as follows

$$x := \alpha_1(x)B(0) + \alpha_2(x)C(0) + \alpha_3(x)[B, C](0), \text{ with } |\alpha_i(x)| \leq 1, \ i = 1, 2, 3.$$

We set (similarly to [11])

$$\beta_1(x) := -\frac{\alpha_1(x)}{\|x\|}, \ \beta_2(x) := -\frac{\alpha_2(x)}{\|x\|}, \text{ and } \beta_3(x) := -\frac{|\alpha_3(x)|}{\|x\|}$$

and use the feedback control $u_{\beta(x)}$ on the interval $[0, 4t_x + t_x^2)$. The condition (5) holds true with $\mu = -1$. According to the main result, this control system is STLAS. Moreover, it could be verified directly that the corresponding trajectory $z(x, u_{\beta(x)}, \cdot)$ is defined on $[0, 4t_x + t_x^2]$ and $\bar{z} := z(x, u_{\beta(x)}, 4t_x + t_x^2) = 0$. Choosing different starting points x from Ω, we apply this feedback control. All computer simulations are performed using the computer algebra system Maple V. Some of the corresponding numerical results are presented in the next table:

x_1	x_2	x_3	t_x	$\bar{z}_1$	$\bar{z}_2$	$\bar{z}_3$
$1/3$	$-2/3$	$5/3$	$\sqrt[4]{\frac{10}{3}}$	0	0	-6.10^{-10}
$1/3$	$2/3$	$-5/3$	$\sqrt[4]{\frac{10}{3}}$	0	-3.10^{-10}	-5.10^{-10}
1	1	-2	$\sqrt[4]{6}$	-3.10^{-10}	0	16.10^{-10}
1	-1	2	$\sqrt[4]{6}$	0	0	0

REFERENCES

[1] R. Bianchini, G. Stefani. Time optimal problem and time optimal map, *Rend. Sem. Mat. Univers. Politecn. Torino* **48** (1990), 401–429.

[2] R. Brockett. Assymptotic stability and feedback stabilization Differential Geometric Control Theory 1983, Progr. Math. **27**, R. Brockett, R. Millmann, H. Sussmann eds., Birkhäuser, Basel-Boston, 181–191.

[3] P. Brunovsky. Local controllability of odd systems, *Banach Center Publications* **1** (1974), Warsaw, 39–45.

[4] F. Clarke. Optimization and nonsmooth analysis., John Willey & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1983.

[5] F. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski. Nonsmooth analysis and control theory, Graduate Text in Mathematics **178** Springer-Verlag, New York, 1998.

[6] F. Clarke, Yu. Ledyaev, L. Rifford, R. Stern. Feedback stabilization and Lyapunov functions, *SIAM Journal on Control and Optimization* **39** (2000), 25-48.

[7] J.-M. Coron. Links between local controllability and local continuous stabilization, IFAC Nonlinear Control Systems Design 1992, M. Fliess ed., Bordeaux, 165-171.

[8] J.-M. Coron. Stabilization in finite time of locally controllable systems by means of continuous time-varying feedback laws, *SIAM J. Control Optim.* **. 33** (1995), 804-833.

[9] H. Frankowska. Local controllability of control systems with feedback **60** (1989), *J. Opt. Theory Appl.*, 277-296.

[10] H. Hermes. Lie algebras of vector fields and local approximation of attainable sets, *SIAM Journal on Control and Optimization* **18** (1980), 352-361.

[11] H. Hermes. On the synthesis of a stabilizing feedback control via Lie algebraic methods, *SIAM Journal on Control and Optimization* **16** (1978), 715-727.
[12] R. Hirshorn. Strong controllability of nonlinear systems, *SIAM J. Control Optim.* **16**, No. 2, 264–275.
[13] V. Jurdjevic, I. Kupka. Polynomial Control Systems, *Mathematische Annalen* **272** (1985), 361-368.
[14] M. Krastanov, M. Quincampoix. Local small-time controllability and attainability of a set for nonlinear control system, *ESAIM: Control Optim. Calc. Var.* **6** (2001), 499-516
[15] H. Kunita. On the controllability of nonlinear systems with application to polynomial systems, *Appl. Math. Optim.* **5** (1979), No. 2, 89–99.
[16] E. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Texts in Applied Mathematics 6, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong (1990).
[17] H. Sussmann. *A general theorem on local controllability*, SIAM *Journal on Control and Optimization* **25** (1987), 158-194.
[18] V. Veliov. On the controllability of control constrained linear systems, *Mathematica Balkanica, New Series* **2** (1988), 147-155.
[19] V. Veliov, M. Krastanov. Controllability of piecewise linear systems, *Systems & Control Letters* **7** (1986), 335-341.

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СТАБИЛИЗИРАЩИ ОБРАТНИ ВРЪЗКИ

Михаил И. Кръстанов

Р. Брокет доказва, че (за разлика от линейния случай) съществуват локално управляеми системи за малко време, които не могат да бъдат стабилизирани с непрекъсната обратна връзка. Следвайки една идея на Х. Хермес, в тази работа е предложен подход за конструиране на прекъсната обратна връзка в околност на затворено множество.