МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2002
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2002
Proceedings of Thirty First Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 3–6, 2002

SOME REQUIREMENTS REGARDING DIFFERENCE SCHEMES FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Rossitza Marinova*, **Hideaki Aiso**, **Tadayasu Takahashi**

In this work we summarize some analytical requirements necessary to be satisfied of difference schemes for incompressible Navier-Stokes equations. The conservation properties of employed approximations of the differential operators which are used in a new vectorial operator splitting scheme for solving the Navier-Stokes equations are discussed. It is proven that the difference approximation of the advection operator conserves square of velocity components and the kinetic energy like the differential operator does, while pressure term conserves only the kinetic energy. **AMS subject classification:** Incompressible Navier-Stokes; Analytical Requirements; Stability and Convergence of Difference Schemes.

1. Introduction. The most important problem is how to construct convergent difference scheme. Since the convergence is a consequence of consistency and stability thus it is necessary to choose those approximating schemes that are stable. It is naturally to have stability in the norms of the spaces for which the original problem is stable. For the well-posed problems of mathematical physics these are the energy spaces where the squares of the norms express the total energy of the systems. Because of this, we have to analyze the derivation of the energy estimations in the differential case and to construct the scheme for which we can satisfy this derivation in the corresponding Hilbert space in the discrete case. However, the criteria of consistency and stability become complicated when applied to the solution of nonlinear partial differential equations. Therefore, the difference scheme has to be conservative, namely, its conservation laws to be satisfied identically. Then the non-linearity is not invincible task. According to [6] the conservation properties of the mass, momentum, and kinetic energy equations for incompressible flow are specified as analytical requirements for a proper set of discrete equations. In present work we summarize some of the analytical requirements necessary to be satisfied of the difference scheme. For illustration the vectorial operator splitting numerical scheme is examined for its conservation properties and other requirements.

2. Incompressible Navier-Stokes equations. Consider the momentum equation and the continuity equation

$$\frac{\partial \mathbf{u}}{\partial t} + C[\mathbf{u}] + P[\mathbf{u}] - V[\mathbf{u}] = 0, \quad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

*The support for R. Marinova from Japan Science and Technology Agency under STA Fellowship ID No.200131 is gratefully acknowledged.

in a closed domain Ω with a piecewise smooth boundary $\partial\Omega$. Here $\mathbf{x} = (x, y, z) \in \Omega$ are Cartesian coordinates, $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the velocity vector,

$$P[\mathbf{u}] = (P_x[u], P_y[v], P_z[w]) = \nabla p,$$

where $p = p(\mathbf{x}, t)$ is the pressure. In the equation (1) the operator $C = C_x + C_y + C_z$ is a short-hand notation for the advection term. Viscous term V is $V = \Delta/Re$, where the Reynolds number is defined as $Re = UL/\nu$, where U is the characteristic velocity, L – characteristic length, ν – kinematic coefficient of viscosity.

In our consideration we assume divergence free initial condition and Dirichlet boundary conditions for velocity, namely

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}\big|_{\partial\Omega} = \mathbf{u}_b. \tag{2}$$

3. Analytical requirements. At first, let us introduce the Hilbert space $H(\Omega)$ of vector-functions with scalar product

$$(\alpha, \beta) = \sum_i (\alpha_i, \beta_i), \quad (\alpha_i, \beta_i) = \int_\Omega \alpha_i(\mathbf{x})\beta_i(\mathbf{x})\mathrm{d}\mathbf{x} \tag{3}$$

and the corresponding norm $\|\alpha\| = (\alpha, \alpha)^{1/2}$, which will be used later.

We summarize the following analytical requirements necessary to be satisfied of the difference scheme:

1. *Conservation properties:* According to [6] we introduce

   Definition 1. *The term $T(\varphi)$ is conservative if it can be written in divergence form*

   $$T[\cdot] = \nabla \cdot (S[\cdot]) \tag{4}$$

   and it is well known that

   (a) The *mass* is conserved a 'priori' since the continuity equation $\nabla\cdot\mathbf{u} = 0$ appears in divergence form.

   (b) The *momentum* is conserved 'a priori' if the continuity equation is satisfied: pressure and viscous terms are conservative 'a priori'; the convective term is also conservative 'a priori' if $\nabla \cdot \mathbf{u} = 0$.

   (c) *Square of a velocity component φ^2:* The advection operator conserves φ^2 if a skew-symmetric form

   $$C_x[\varphi] = \frac{1}{2}\left[\frac{\partial(\varphi u)}{\partial x} + u\frac{\partial\varphi}{\partial x}\right], \quad C_y[\varphi] = \frac{1}{2}\left[\frac{\partial(\varphi v)}{\partial y} + v\frac{\partial\varphi}{\partial y}\right], \quad \text{etc.} \tag{5}$$

   is used. Here φ is one of the velocity components u, v, and w. For instance, in the direction x we have

   $$\varphi\, C_x[\varphi] = \frac{\varphi}{2}\left[\frac{\partial(\varphi u)}{\partial x} + u\frac{\partial\varphi}{\partial x}\right] = \frac{1}{2}\frac{\partial(\varphi^2 u)}{\partial x}. \tag{6}$$

   Hence, the operator C_x is conserving square of a velocity component φ^2. It means that under the assumption of homogenous boundary conditions we have

   $$(C_x[\varphi], \varphi) = (C_y[\varphi], \varphi) = (C_z[\varphi], \varphi) = 0 \text{ or } (C[\varphi], \varphi) = 0. \tag{7}$$

The pressure term in the momentum equation is not conservative, since

$$u\frac{\partial p}{\partial x} = \frac{\partial (up)}{\partial x} - p\frac{\partial u}{\partial x} \tag{8}$$

for the velocity component u, for instance.

Similarly, the viscous term in the equation for u satisfies

$$u\Delta u = u\nabla^2 u = \nabla \cdot (u\nabla u) - (\nabla u)^2 \tag{9}$$

or, in other words, it is not conservative as well.

(d) *Kinetic energy* $K \overset{\text{def}}{=} \frac{1}{2}(u^2 + v^2 + w^2)$: It follows from (7) that the operator $C[\mathbf{u}] = \frac{1}{2}\mathbf{u}[\nabla \cdot (\mathbf{uu}) + \mathbf{u} \cdot \nabla \mathbf{u}]$ conserves the kinetic energy K, i.e.

$$\mathbf{u} \cdot C[\mathbf{u}] = \frac{1}{2}\mathbf{u} \cdot [\nabla \cdot (\mathbf{uu}) + \mathbf{u} \cdot \nabla \mathbf{u}] = \frac{1}{2}[\nabla \cdot (\mathbf{uu}^2)]. \tag{10}$$

The pressure term is energy conservative if the continuity equation is satisfied

$$\mathbf{u} \cdot \nabla p = \nabla \cdot (\mathbf{u}p) - p(\nabla \cdot \mathbf{u}) = \nabla \cdot (\mathbf{u}p), \tag{11}$$

while the viscous term is not conservative

$$\mathbf{u} \cdot \Delta \mathbf{u} = \mathbf{u} \cdot \nabla^2 \mathbf{u} = \nabla \cdot (\mathbf{u}\nabla \mathbf{u}) - (\nabla \mathbf{u})^2 \tag{12}$$

because of the kinetic energy dissipation – the second term in the right-hand side of (12).

2. *Compatibility condition for Poisson equation for pressure*, see [1, 7], should be satisfied if the numerical method uses a Poisson equation for pressure instead of the continuity equation

$$\int_\Omega F_p \mathrm{d}\mathbf{x} = \oint_{\partial\Omega} \frac{\partial p}{\partial n}\mathrm{d}s,$$

where F_p is right hand side for pressure equation, n is the outward normal to the boundary contour $\partial\Omega$.

3. *Commutativity* of Laplacian operator Δ and divergence operator ∇.

4. *Consistency* between gradient and divergence operators

$$\int_\Omega [\mathbf{u} \cdot \nabla p + p(\nabla \cdot \mathbf{u})]\mathrm{d}\mathbf{x} = \oint_{\partial\Omega} p v_n \mathrm{d}s$$

should be satisfied as well. For instance, the consistency is necessary in order to obtain skew-symmetric operator P. The mutually consistent discretizations of operators gradient and divergence with a first-order truncation error on non-staggered grids are derived in [7]. In the case of such grids the use of standard central difference approximation for gradient leads to the same approximation for divergence and yields for the discretization of pressure Laplace operator an extended stencil and checker-board effect. On staggered grid the consistency between gradient and divergence operators is not difficult to be satisfied.

The satisfaction of (1)–(5) leads to strong L_2 stability of the scheme. Therefore, the purpose is to derive difference scheme that satisfies the above requirements in a discrete sense.

4. Difference operators and their properties. For the case under consideration the flow occupies the region with rectilinear boundaries in cartesian coordinates. The grid is *staggered* for u in x-direction, for v in y-direction, and for w in z-direction. For boundary conditions involving derivatives this allows one to use central differences with second-order of approximation on two-point stencils. The number of main grid lines (which are, in fact, the p-grid lines) in the x-, y- and z-directions are respectively N_x, N_y, and N_z. The coordinates of the grid points are (x_i, y_j, z_k) for $i = 1, \ldots, N_x$, $j = 1, \ldots, N_y$, $k = 1, \ldots, N_z$. The spacings are given by $h^p_{x,i} = x_{i+1} - x_i$, $i = 1, \ldots, N_x - 1$, $h^p_{y,j} = y_{j+1} - y_j$, $j = 1, \ldots, N_y - 1$, and $h^p_{z,k} = z_{k+1} - z_k$, $k = 1, \ldots, N_z - 1$. The spacings for the function u in direction x are defined as follows

$$h^u_{x,1} = h^p_{x,1},\ h^u_{x,i} = (h^p_{x,i} + h^p_{x,i-1})/2 \text{ for } i = 2, \ldots, N_x - 1, \text{ and } h^u_{x,N_x} = h^p_{x,N_x-1}.$$

Similarly the spacings for v in direction y and for w in direction z are defined

$$h^v_{y,1} = h^p_{y,1},\ h^v_{y,j} = (h^p_{y,j} + h^p_{y,j-1})/2 \text{ for } j = 2, \ldots, N_y - 1\ h^v_{y,N_y} = h^p_{y,N_y-1}, \tag{13}$$

$$h^w_{z,1} = h^p_{z,1},\ h^w_{z,k} = (h^p_{z,k} + h^p_{z,k-1})/2 \text{ for } k = 2, \ldots, N_z - 1\ h^w_{z,N_z} = h^p_{z,N_z-1}. \tag{14}$$

The pressure is sampled at the points labelled by "•"; function u – at "∘"; function v – at "∗", and function w – at "⋄". The following notations are used:

$$p_{i,j,k} = p(x_i, y_j, z_k), \quad u_{i,j,k} = u(x_i - h^p_{x,i-1}/2, y_j, z_k), \tag{15}$$

$$v_{i,j,k} = v(x_i, y_j - h^p_{y,j-1}/2, z_k), \quad w_{i,j,k} = w(x_i, y_j, z_k - h^p_{z,k-1}/2). \tag{16}$$

For the second derivatives standard three point difference approximations are employed, which inherit the negative definiteness of the respective differential operators. The first derivatives for pressure at the mesh-point labelled by "∘", "∗", and "⋄" are approximated in the following way:

$$P^h_x[u]\Big|_{\circ} = \frac{p_{i,j,k} - p_{i-1,j,k}}{h^p_{x,i-1}},\ P^h_y[v]\Big|_{*} = \frac{p_{i,j,k} - p_{i,j-1,k}}{h^p_{y,j-1}},\ P^h_z[w]\Big|_{\diamond} = \frac{p_{i,j,k} - p_{i,j,k-1}}{h^p_{z,k-1}}. \tag{17}$$

The skew-symmetric difference approximation of the advection term was proposed by Arakawa [2] for the $\psi - \omega$ formulation for ideal flows. A similar idea in primitive variables was elaborated in [5] with a special reference to the operator-splitting schemes. In [3, 4] we consider second order conservative approximations of the non-linear operators on a uniform staggered mesh. For instance, on a non-uniform staggered mesh, we employ the following conservative approximations for the nonlinear terms in the momentum equation for velocity component u with respect to direction x

$$C^h_x[u] = \left(\frac{\partial(u^2)}{\partial x} - \frac{u}{2}\frac{\partial u}{\partial x}\right)\Big|_{\circ} = \frac{u^m_{i+1/2,j,k}u_{i+1,j,k} - u^m_{i-1/2,j,k}u_{i-1,j,k}}{h^u_{x,i} + h^u_{x,i-1}}, \tag{18}$$

where $u^m_{i+1/2,j,k} = (u^m_{i+1,j,k} + u^m_{i,j,k})/2$, $u^m_{i-1/2,j,k} = (u^m_{i,j,k} + u^m_{i-1,j,k})/2$, etc. The differences for nonlinear terms in the equations for v and w are similar to (18).

It can be proven that the defined approximations of the nonlinear advection terms preserve their skew-symmetric property. The following statement is valid

Lemma 1. *Let appropriate (homogenous, periodic, etc.) boundary conditions are acknowledged and the scalar product is*

$$(\alpha, \beta) \stackrel{\text{def}}{=} \sum_{i,j,k} \alpha_{i,j,k}\beta_{i,j,k}\hbar^f_{x,i}\hbar^f_{y,j}\hbar^f_{z,k}, \tag{19}$$

where $\hbar^f_{x,i} = (h^f_{x,i} + h^f_{x,i-1})/2$, $\hbar^f_{y,j} = (h^f_{y,j} + h^f_{y,j-1})/2$, $\hbar^f_{z,k} = (h^f_{z,k} + h^f_{z,k-1})/2$, *and* $f = u$, v, w, *or* p. *Then the equalities hold true*

$$(C^h_x[f], f) = 0, \quad (C^h_y[f], f) = 0, \quad (C^h_z[f], f) = 0. \tag{20}$$

Hence the defined approximations of the nonlinear terms on a non-uniform staggered grid preserve their skew-symmetric property. It follows immediately that:

Theorem 1. *Under the assumptions of Lemma* 1, *the following relations are satisfied* $(C^h[u], u) = (C^h[v], v) = (C^h[w], w) = 0$.

From the above theorem it follows $(C^h[u], u) + (C^h[v], v) + (C^h[w], w) = 0$, hence

Corollary 1. *The advection term is energy conservative.*

Similarly, it is not difficult to be proven (taking into account the approximation of the divergence operator) that the pressure term approximation conserves the kinetic energy K in the case of uniform grids. Under the assumptions of Lemma 1 the following relation is satisfied

$$(P^h_x[u], u) + (P^h_y[v], v) + (P^h_z[w], w) = 0, \tag{21}$$

and the result can be summarized in the next

Theorem 2. *The pressure term is energy conservative if the grid is uniform.*

The conclusion is that under not so restrictive assumptions the chosen approximations of the differential equations and boundary conditions satisfy the formulated requirements to difference schemes for incompressible flows. The difference scheme is strongly stable in solving higher Reynolds number flows which is demonstrated in [4].

REFERENCES

[1] S. Abdallah. Numerical solutions for the pressure Poisson equation with Neumann boundary condition using a non-staggered grid. Part I. *J. Comput. Phys.*, **70** (1987), 182–192.

[2] A. Arakawa. Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I. *J. Comput. Phys.*, **1** (1966), 119–143.

[3] C. I. Christov, R. S. Marinova. Implicit scheme for Navier-Stokes equations in primitive variables via vectorial operator splitting. *Notes on Numer. Fluid Mech.*, **62** (1998), 251–259.

[4] C. I. Christov, R. S. Marinova. Implicit vectorial operator splitting for incompressible Navier-Stokes equations in primitive variables. *J. Comp. Techn.*, (2001), to appear.

[5] G. I. Marchuk. Methods of numerical mathematics. Berlin, Springer, 1982.

[6] Y. Morinishi, T. Lund, O. Vasiliev, P. Moin. *J. Comp. Phys.*, **143**, (1998), 90-124.

[7] A. Churbanov, A. Pavlov, P. Vabishchevich. Operator-splitting Methods for the Incompressible Navier-Stokes Equations on Non-staggered Grids. Part 1: First Order Schemes. *Int. J. Numer. Meth. Fluids*, **21** (1995), 617–640.

[8] N. N. Yanenko. Method of Fractional Steps. Berlin, Springer, 1971.

Rossitza Marinova
CFD Technology Center
National Aerospace Laboratory
Jindaiji-Higashi-machi 7-44-1 Chofu
Tokyo 182-8522 Japan
e-mail: `marinova@nal.go.jp`

Hideaki Aiso
CFD Technology Center
National Aerospace Laboratory
Jindaiji-Higasi-machi 7-44-1 Chofu
Tokyo 182-8522 Japan
e-mail: `aiso@nal.go.jp`

Tadayasu Takahashi
CFD Technology Center
National Aerospace Laboratory
Jindaiji-Higashi-machi 7-44-1 Chofu
Tokyo 182-8522 Japan
e-mail: `takahasi@nal.go.jp`

НЯКОИ ИЗИСКВАНИЯ ОТНОСНО ДИФЕРЕНЧНИ СХЕМИ ЗА НЕСВИВАЕМИ УРАВНЕНИЯ НА НАВИЕ-СТОКС

Росица Маринова, Хидеаки Айзо, Тадаясуі Таканаши

В тази работа са обобщени някои аналитични изисквания, които диференчните схеми за несвиваемите уравнения на Навие-Стокс е необходимо да удовлетворяват. Обсъдени са консервативните свойства на апроксимациите на диференциалните оператори, които са използвани в една нова схема на векторно разцепване на оператора за решаване на уравненията на Навие-Стокс. Доказано е, че диференчната апроксимация на конвектния оператор запазва консервативните свойства на квадрата на компонентите на скоростта и кинетичната енергия, както това е изпълнено за диференциалния оператор, докато членът на налягането запазва само кинетичната енергия.