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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Apostol Metodiev Poceski

The method is based on the approximation of the derivatives within a small segment of the region of integration by a product of lower order derivatives of the interpolation function (IF). This function has to represent solution of the homogeneous portion, or solution of the highest order derivative term of the differential equation. The method represents extension of the finite element and finite difference methods, to the solution of many problems defined by differential equations. The application of the method is illustrated on the solution of differential equations of the boundary value problems. The solution of the differential equation of vibration (as a fourth order one -initial value problem), is presented in more details. Much improved numerical results are derived, particularly in the case of damped vibrations.

1. Introduction. One of the most popular numerical methods was the finite difference method (FDM). However, in the recent decades the most popular method of solution of many engineering problems became the finite element method (FEM). This method has advantage over the FDM, primarily in the successful simulation of arbitrary boundary conditions. A numerical method, with a mathematical approach, would have some advantages over the standard FEM. Such is, for instance, derivation of finite elements on the base of the differential equations [3], or FEM based on the differential equations [4].

2. Approximation of the derivatives. For derivation of a numerical solution at certain discrete points the derivatives of the governing equation can be approximated as follows:

$$\frac{d^2u}{dx^2} = -\frac{d\phi}{dx}\cdot\frac{du}{dx}\ ;\ \frac{d^4u}{dx^4} = \frac{d^2\phi}{dx^2}\cdot\frac{d^2u}{dx^2}\ ;\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial\phi}{\partial x}\cdot\frac{\partial u}{\partial y} \text{ etc.} \tag{1}$$

$i-1$ - - - h_1 - - - - I $\quad$ i - - - h_2 - - - $i+1$
o- - - - - - - - - - - - - - - - -o $\quad$ o- - - - - - - - - - - - - - - - -o
1 $\quad -1$ $\quad -1$ $\quad$ 1

Fig. 1. Second and fourth finite differences
as independent contribution of two segments

where $u = \phi u_i$, ϕ is IF, u_i – vector of the segment nodal parameters. The second finite difference can be represented as independent contribution of two segments, Fig. 1, as follows:

$$\frac{d^2u}{dx^2} \approx \frac{u_{i-1} - u_i}{h_1^2} + \frac{u_{i+1} - u_i}{h_2^2} \tag{2}$$

The expression for the fourth finite difference $\dfrac{d^4u}{dx^4}$ will be same, but in terms of the second derivatives $\dfrac{d^2u}{dx^2}$. It yields the segment matrix (element matrix) as follows,

$$\frac{d^2u}{dx^2} = \frac{1}{h^2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \tag{3}$$

In a case of a second order differential equation like,

$$\frac{d^2u}{dx^2} = -m(x) \tag{4}$$

the IF has to be solution of the homogeneous portion of this equation:

$$\iint \frac{d^2y}{dx^2}dx = c_1 + c_2x \tag{5}$$

The integration constants are expressed in terms of the nodal solutions u_i and u_{i+1}. The substitution of the approximation (1a) into (4) and integration of it yields,

$$\int_0^h \frac{d\phi^T}{dx} \cdot \frac{du}{dx}dx = \frac{h}{h^2}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix} \tag{6}$$

This means that approximation of the derivative (1) yields exactly the second finite difference (3), multiplied.by segment lenght h (due to integration).The integration of the right hand side of (4) yields total contribution. However, in order to get contribution to the two rows of (6) separately, this side has to be multiplied by ϕ^T. So, the solution of (4) will be,

$$\int_0^h \frac{d\phi^T}{dx} \cdot \frac{du}{dx}dx = \int_0^h \phi^T m(x)dx \tag{7}$$

This expression yields is a 2×2 matrix equation, representing contribution of the segment solution to the two system equations at the nodes (Fig. 1). The right hand side, for instance in the case of $m(x) =$ constant, yields $hm/2$. The system equations derived on the base of this soltion, by application of the standard FEM procedure are the same as the FDM equations, both sides multiplied by h. Therefore, regardless of the number of subdivisions and $m(x)$, exp.(7) **always yields exact results**. In the case of fourth order differential equation,

$$\frac{d^4u}{dx^2} = p(x) \tag{8}$$

the same procedure yields the following solution,

$$\int_0^h \frac{d^2\phi^T}{dx^2} \cdot \frac{d^2u}{dx^2}dx = \left(\frac{d^3u}{dx^3}\right)_i = h\frac{\Delta^4 u}{\Delta x^4} = \int_0^h \phi^T p(x)dx \tag{9}$$

where ϕ in this case is a third order polynomial. One should note that the segment matrix can be derived directly, as third derivative of the interpolation function at the nodes (i), which is equivalent to the fourth finite difference multiplied by the segment lenght h. This solution yields **exact results also**. Similar expression has been applied in the beam bending problem for computation of deflection at certain points, since long ago (Mohr-Maxwell method).

2. Solution of some differential equations. The application of the method will be illustrated on the solution of engineering problems, but the method readily can be applied on solution of biomechanics problems. The *buckling* of a compressed beam is governed by the following differential equation,

$$\mathrm{EI}\frac{d^2u}{dx^2} + Nu = -m(x) \tag{10}$$

A problem of solution of this type of equations represents the case when this is a *matrix equation*. Therefore a numerical solution has to be applied. The solution of this equation requires a linear IF (a very rough one). In order to improve the accuracy with less segments of subdivision, it is derivated twice, and the following derived,

$$\mathrm{EI}\frac{d^4u}{dx^4} + N\frac{d^2u}{dx^2} = p(x) \tag{11}$$

The solution oft his equation is a combination of (7) and (9),

$$\mathrm{EI}\int\limits_0^h \frac{d^2\phi^T}{dx^2}\cdot\frac{d^2u}{dx^2}dx - N\int\limits_0^h \frac{d\phi^T}{dx}\cdot\frac{du}{dx}dx = \int\limits_0^h \phi^T p(x)dx \tag{12}$$

The IF in this case is a third order polynomial,

$$u = \phi u_i = c_1 + c_2x + c_3x^2 + c_4x^3 \tag{13}$$

The coefficients of this expression are defined by the nodal solutions u_i and first derivatives u_i'. This IF is good enough for the first term of (12), but not for the second. That is why eq.12 does not yield exact results. However, the error of the computed critical force N, acting on a beam represented by one segment only, is within 1%!

The *plate bending problem* is governed by the following differential equation,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x,y)/D \tag{14}$$

The best assumption for the IF is a polynomial that satisfies the homogeneous portion. That is the 12 term polynomial as follows,

$$w = c_1 + c_2x + c_3y + c_4x^2 + \cdots + c_{11}x^3y + c_{12}xy^3 \tag{15}$$

where the first 10 terms are full third order polynomial of the Pascal's triangle. The substitution of the approximations of type (1) into (14) yields the following equation for a numerical solution,

$$\iint\left[\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) + 2(1-v)\frac{\partial^2\phi}{\partial x\partial y}\cdot\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)\right]dxdy =$$

$$= \iint \frac{p}{D}\phi^t dxdy \tag{16}$$

In a similar way, the partial differential equations of the plane stress problem, three-dimensional and shell problems, were solved [2–4]. The results of numerical examples show very good accuracy of the derived solutions.

3. Solution of the differential equation of vibration. The *equation of vibration* of a single degree of freedom system is similar to (10), but in terms of time, as follows,

$$m\frac{d^2u}{dt^2}+c\frac{du}{dt}+ku=p(t) \tag{17}$$

The solution of this equation does not represent a problem. However, a problem represents when this is a matrix equation. In order to get an improved numerical solution, a high order IF has to be used. Therefore, this equation has to be differentiated twice, yielding the following,

$$m\frac{d^4u}{dt^4}+c\frac{d^3u}{dt^3}+k\frac{d^2u}{dt^2}=\frac{d^2p}{dt^2} \tag{18}$$

The substitution of approximations of the type (1) yields,

$$m\int\frac{d^2\phi}{d^2t}\cdot\frac{d^2u}{dt^2}dt+c\int\frac{d^2\phi}{d^2t}\cdot\frac{du}{dt}dt-k\int\frac{d\phi}{dt}\cdot\frac{du}{dt}dt=-\int\frac{d\phi_p}{dt}\cdot\frac{dp}{dt}dt \tag{19}$$

The IF ϕ in this case is similar to (13) – a third order polynomial. The coefficients of the polynomial are expressed in terms of the solutions u and first derivative u' at the beginning and at the end of the step of integration. The solution, in increments during the short time step (segment) of integration, is as follows,

$$\Delta u=\frac{\Delta p\alpha_1+(6m/\Delta t)u_1'\alpha_2+3m\ddot{u}_1\alpha_3}{6m/\Delta t^2+(3c/\Delta t)\alpha_4+k\alpha_5}$$

$$\Delta u'=\frac{3\Delta p\alpha_3/\Delta t+mu_1'\alpha_6/\Delta t^2+6m\ddot{u}_1\alpha_2'/\Delta t}{6m/\Delta t^2+(3c/\Delta t)\alpha_4+k\alpha_5} \tag{20}$$

The coefficients in these expressions are as follows,

$$\begin{aligned}&\alpha_1=1+\Delta t^2k/60m+c\Delta t/4m; && \alpha_2=1-0.1\Delta t^2k/m+c\Delta t/2m\\ &\alpha_2'=1-0.1\Delta t^2k/m; && \alpha_3=1-\Delta t^2k/60m+c\Delta t/6m;\\ &\alpha_4=1-\Delta t^2k/15m+c\Delta t/6m; && \alpha_5=0.4+\Delta t^2k/40m;\\ &\alpha_6=-3\Delta t^2k/m\end{aligned} \tag{21}$$

The solution of the linear acceleration method (LAM) is as follows,

$$\Delta u=\frac{\Delta p+m(6u_1'/\Delta t+3\ddot{u}_1)+c(3u_1'+\ddot{u}_1\Delta t/2)}{6m/\Delta t^2+3c/\Delta t+k} \tag{22}$$

This solution seems to be the best available at present, at least for analysis of SDFS, and therefore it is taken for comparison. The increment of the first derivative in LAM is computed as follows,

$$\Delta u'=3\Delta u/\Delta t-u_i'-\ddot{u}_i\Delta t/2 \tag{23}$$

By adding the increments, the values at the end of the step are derived,

$$u_{i+1}=u_i+\Delta u\ ;\ \ u_{i+1}'=u_i'+\Delta u'$$

In the application of both methods the second derivative has to be computed on the base of equilibrium equation (17),

$$\ddot{u}_{i+1}=p(t)/m-u_{i+1}k/m-u_{i+1}'c/m \tag{24}$$

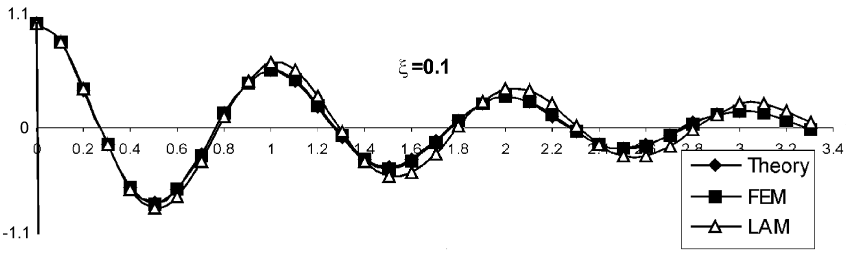

Fig.2 Free vibration test, the results of the FEM solution (presented method) practically are the same as the theoretical, while the LAM results are with period elongation and an error at the end of the third cycle of almost 50%.

These are initial values for the next step of integration.

Some results derived recently, by application of eqs 20–21 (FEM) and eq. 22 (LAM) are presented on Fig. 2. The results show superiority of the FEM solution over the LAM solution, which, otherwise, has been considered as the best one. The accuracy of this solution is the best available at present. The FEM results are almost the same as theoretical, with errors of small percentage, while the LAM results at the end of the third cycle are with error of almost 50% !

The FEM solution presented here can be applied on the analysis of brain functioning,for instance to study the neutrons. The governing first order differential equation has to be differentiated once and the displacements as unknown nodal parameters to be taken.

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Apostol Metodiev Poceski
Gradezen Fakultet, Skopje
R. Macedonia
e-mail: apoce@ukim.edu.mk

ЕДИН ЧИСЛЕН МЕТОД ЗА РЕШАВАНЕ НА ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ

Апостол Посецки

Методът се основава на апроксимация на производните в малка подобласт от областта на интегриране с производни от по-нисък ред на интерполационната функция (ИФ). Тази функция е решение на хомогенната част на уравнението или на тази част от него, съдържаща производната от най-висок ред. Методът представлява разширение на метода на крайните елементи и на крайните разлики за числено решаване на диференциални уравнения. Прилагането му е илюстрирано с решаване на една гранична задача (уравнение на вибрации).