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NONTRIVIAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SEMILINEAR FOURTH AND SIXTH ORDER DIFFERENTIAL EQUATIONS*

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The existence of nontrivial solutions of boundary value problems for semilinear fourth and sixth order differential equations arising in phase transition models is discussed via variational methods.

1. Introduction. Bistable systems play an important role in the study of spatial patterns. A typical example, which appears in the population dynamics, leads to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3,$$

referred to as the classical Fisher-Kolmogorov (FK) equation (cf. [5]). The term "bistable" refers here to the fact the stationary states $u_{1,2} = \pm 1$ are stable as solutions of the equation $\frac{du}{dt} = u - u^3$.

Recently, an interest to the higher-order extension of the FK equation, known as extended FK (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \gamma > 0$$

has shown.

It was proposed by Coullet, Elphick & Repaux [2] and Dee & van Saarloos [3]. With the change of variables

$$p = (\gamma)^{-1/2}, \ u(x) = \overline{u}\left(\gamma^{1/4} x\right),$$

the stationary EFK-equation

$$-\gamma \overline{u}^{iv}(x) + \overline{u}''(x) + \overline{u}(x) - \overline{u}^3(x) = 0$$

turns to

$$u^{iv}(x) - pu''(x) - u(x) + u^3(x) = 0. \tag{1}$$

The EFK equation has appeared in the studies of phase transition, for instance near a Lifshitz point and in the studies of spatial and temporal pattern formation in bistable systems (cf. Zimmerman [13], Dee & van Saarloos [3]). The bounded solutions of the

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equation (1) were extensively studied by Peletier, Troy & van der Vorst [6], Peletier & Troy [7], [8] applying variety of methods as topological shooting methods, phase-plane analysis and variational methods. Phase field models involving higher-order spatial gradients can also lead to EFK equations. We mention the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^4 u}{\partial x^4} + u - u^3,$$

studied in Gardner & Jones [4].

In this paper we study the periodic solutions of the fourth-order equations

$$(I) \qquad u^{iv} - pu'' - a(x)\,u + b(x)\,u^3 = 0,$$

as well as of the sixth-order equations

$$(II) \qquad u^{vi} + Au^{iv} + Bu'' + u - b(x)\,u^3 = 0,$$

where $p \neq 0$ is a constant, a and b are even continuous positive $2L-$ periodic functions, and the constants A and B satisfy $A^2 < 4B$. Note that if $p > 0$, Eq.(I) is referred to as EFK-equation, while if $p < 0$ Eq.(I) is referred to as extended Swift–Hohenberg (ESH)-equation.

Denote by (P_1) the boundary value problem (BVP) for Eq.(I) with the boundary conditions

$$u(0) = u(L) = u''(0) = u''(L) = 0,$$

and by (P_2) the BVP for Eq.(II) subject the boundary conditions

$$\begin{aligned} u(0) &= u''(0) = u^{iv}(0) = 0, \\ u(L) &= u''(L) = u^{iv}(L) = 0. \end{aligned}$$

We obtain $2L$-periodic solutions which are antisymmetric with respect to 0 and L taking the $2L$-periodic extension of the odd extension of solutions u of the problems to $[-L, L]$.

The problem (P_1) was studied in recent works of Tersian & Chaparova [11], [1] while (P_2) was studied in [12] using variational methods.

The problem (P_1) has a variational structure and its weak solutions in the space $X(L) := H^2(0,L) \cap H_0^1(0,L)$ can be found as critical points of the functional

$$I(u;L) := \int_0^L \left(\frac{1}{2}\left(u''^2 + pu'^2 - a(x)\,u^2\right) + \frac{1}{4}b(x)\,u^4\right) dx.$$

The problem (P_2) has also a variational structure and its weak solution can be found as a critical point of the functional

$$J(u;L) := \int_0^L \left(\frac{1}{2}\left(u'''^2 - Au''^2 + Bu'^2 - u^2\right) + \frac{1}{4}b(x)\,u^4\right) dx.$$

in the space

$$Y(L) := \{u \in H^3(0,L) : u(0) = u''(0) = u(L) = u''(L) = 0\}.$$

The critical points of the functionals I and J are found using the Clark's theorem (cf. Rabinowitz [9], Theorem 9.1).

Theorem (Clark). *Let X be a real Banach space, $E \in C^1(X, \mathbf{R})$ with E even, bounded from below, and satisfying (PS) condition. Suppose that $E(0) = 0$ and there is a set $K \subset X$ such that K is homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and $\sup_K E < 0$. Then E possesses at least m distinct pairs of critical points.*

We collect the results, found in [1], [11], [12] in the following

Theorem A. *Let $p \neq 0, a$ and b be positive continuous functions on $[0, L]$ and A, B be constants such that $A^2 < 4B$. Then there exists numbers $L_1, L_2, 0 < L_2 < L_1$ such that the problems (P_1) and (P_2) have only the trivial solution if $0 < L \leq L_2$ and at least m distinct pairs of solutions if $L > mL_1$ for $m \in \mathbf{N}$.*

It is a natural question what happens when $L \in (L_2, L_1)$? The answer to the question is based on the abstract Theorem 1.1 for functionals in a Hilbert space H of the form $E(u) = a(u, u) - b(u, u) + \psi(u)$, where a and b are positive bilinear forms on $H \times H$ and ψ is a convex functional.

Our second main result is

Theorem B. *Suppose that the assumptions of Theorem A hold. There exists a number $L^* \in (L_2, L_1)$ such that the problem (P_1) has the trivial solution if $0 < L \leq L^*$ and a nontrivial solution if $L > L^*$. The same holds for the problem (P_2) in case $A = 0$.*

The paper is organized as follows. In Section 2 we consider the solvability of problem (P_1), also we formulate an abstract result for functionals in Hilbert spaces. In Section 3 we deal with the problem (P_2).

2. Solvability of problem (P_1). First we study the solvability of the problem (P_1) where p, a and b satisfy

$$p \neq 0, 0 < a_1 \leq a(x) \leq a_2, 0 < b_1 \leq b(x) \leq b_2. \tag{2}$$

Let $X(L) = H^2(0, L) \cap H_0^1(0, L)$ be the Hilbert space with the inner product

$$(u, v) = \int_0^L (u''v'' + u'v' + uv)\, dx.$$

A function $u \in X(L)$ is said to be a weak solution of (P_1) if for arbitrary function $v \in X$

$$\int_0^L \left(u''v'' + pu'v' - a(x)\, uv + b(x)\, u^3 v\right) dx = 0.$$

The weak solutions of (P_1) are critical points of the functional $I : X \to \mathbf{R}$

$$I(u) = \frac{1}{2}\int_0^L \left(u''^2 + pu'^2 - a(x)\, u^2\right) dx + \frac{1}{4}\int_0^L b(x)\, u^4 dx.$$

A weak solution $u \in X(L)$ of (P_1) is also a classical solution of (P_1), cf. [11]. The proof of Theorem A for the problem (P_1) is divided in a sequence of lemmata, proved in [1], [11].

Let

$$L_j := \pi\sqrt{\frac{p + \sqrt{p^2 + 4a_j}}{2a_j}}, \quad j = 1, 2. \tag{3}$$

We have:

Lemma 2.1. *The functional I is bounded from below on $X(L)$, I has a bounded minimizing sequence and there exists a minimizer u of the functional I.*

Lemma 2.2. *If $L \leq L_2$, then the problem (P_1) has only the trivial solution.*

Lemma 2.3. *The functional I satisfies the (PS) condition.*

Lemma 2.4. *Let $L > mL_1$ for some $m \in \mathbf{N}$. Then there exists a set $K \subset X(L)$ homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and $\sup_K I < 0$.*

Now we discusse the solvability of the problem (P_j) when $L_2 < L < L_1$. First we prove an abstract theorem for functionals on Hilbert spaces.

Let H be a Hilbert space, $E : H \to \mathbf{R}$ be a functional of the form

$$E(u) = a(u,u) - b(u,u) + \psi(u), \tag{4}$$

where the bilinear forms a and b satisfy the assumption

$$(A_1): \begin{cases} a : H \times H \to \mathbf{R} \text{ be a scalar product in } H, \\ b : H \times H \to \mathbf{R} \text{ be a symmetric bilinear form, } b(u,u) > 0, u \neq 0, \\ b - \text{weakly continuous, i.e. if } u_k \rightharpoonup u \Rightarrow b(u_k,u_k) \to b(u,u). \end{cases}$$

Suppose that ψ satisfies

$$(A_2): \begin{cases} \psi \in C^1(H, \mathbf{R}) \text{ is a convex functional,} \\ \psi(u) > 0, u \neq 0 \text{ and } \psi(0) = 0. \end{cases}$$

It is known that under assumption (A_1) (cf. [10, p.29]) the eigenvalue problem $a(u,v) = \lambda\, b(u,v)$, $\forall v \in H$ has a countable set of positive eigenvalues and eigenvectors (λ_n, e_n). The first positive eigenvalue is characterized as

$$\lambda_1 = \min_{u \in H \setminus \{0\}} \frac{a(u,v)}{b(u,v)}. \tag{5}$$

We have

Theorem 2.1. *Let bilinear forms a and b on Hilbert space H satisfy (A_1) and $\lambda_1 > 1$. Let $\psi : H \to \mathbf{R}$ satisfy assumption (A_2). Then the functional $E(u) = a(u,u) - b(u,u) + \psi(u)$ has only the trivial critical point. Let $0 < \lambda_1 < 1$ and ψ satisfy moreover $\psi(tu) = |t|^k \psi(u), k > 2$. If the functional E has a bounded minimizing sequence then E has a nontrivial critical point.*

Let us consider the problem (P_1) and two cases $p > 0$ and $p < 0$. Suppose that $p > 0$ and make a change of variables, $t = \frac{x}{L}, u(t) = u\left(\frac{x}{L}\right)$. The problem (P_1) is equivalent to the problem

$$(P_1'): \begin{cases} u^{iv} - L^2 p u'' - L^4 a u + L^4 b u^3 = 0, 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

and its weak solutions are critical points of the functional

$$I_1(u) = \frac{1}{2} \int_0^1 \left(u''^2 + L^2 p u'^2 - L^4 a u^2\right) dt + \frac{1}{4} \int_0^1 b u^4 dt$$

on the space $X_1 = X(1)$. The functional I_1 can be represented in the form

(6) $$I_1(u) = a_1(u,u) - b_1(u,u) + \psi_1(u),$$

where

$$a_1(u,v) = \frac{1}{2}\int_0^1 \left(u''v'' + L^2pu'v'\right)dt$$

and

$$b_1(u,v) = \frac{1}{2}L^4\int_0^1 au^2dt, \quad \psi_1(u,L) = \frac{1}{4}L^4\int_0^1 bu^4dt.$$

The forms a_1 and b_1 are symmetric positive bilinear forms, which satisfy (A_1) in X_1, $\psi_1(u)$ is a positive convex functional and there exists

$$\lambda_1(L) = \min_{u\in X_1\setminus\{0\}} \frac{\int_0^1 \left(u''^2 + L^2pu'^2\right)dx}{L^4\int_0^1 au^2dx} > 0.$$

Lemma 2.5. *There exists $L_1^* > 0$ such that $\lambda_1(L_1^*) = 1$ and*

$$(\lambda_1(L) - 1)(L - L_1^*) < 0, \quad L > 0.$$

In view of Theorem 2.1, Lemma 2.1 and Lemma 2.5 we have

Theorem 2.2. *There exists a number $L_1^* \in (L_2, L_1)$, where $L_j, j = 1,2$ are given by (3) such that the problem (P_1) with $p > 0$ has only the trivial solution if $0 < L \leq L_1^*$ and a nontrivial solution if $L > L_1^*$.*

Suppose that $p < 0, p = -q, q > 0$ and make a change of variables, $t = \frac{x}{L}$, $u(t) = u\left(\frac{x}{L}\right)$. The problem (P_1) is equivalent to the problem

$$(P_2') : \begin{cases} u^{iv} + L^2qu'' - L^4au + L^4bu^3 = 0, 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

and its weak solutions are critical points of the functional

$$I_2(u) = \frac{1}{2}\int_0^1 \left(u''^2 - L^2qu'^2 - L^4au^2\right)dt + \frac{1}{4}\int_0^1 bu^4dt$$

The functional I_2 can be represented in the form

$$I_2(u) = a_2(u,u) - b_2(u,u) + \psi_2(u),$$

where

$$a_2(u,v) = \frac{1}{2}\int_0^1 u''v''dt$$

and

$$\begin{aligned} b_2(u,v) &= \frac{1}{2}\int_0^1 \left(L^2qu'v' + L^4auv\right)dt, \\ \psi_2(u) &= \frac{1}{4}L^4\int_0^1 bu^4dt. \end{aligned}$$

The forms a_2 and b_2 are symmetric positive bilinear forms, which satisfy (A_1) in X_1, $\psi_2(u)$ is a positive convex functional and there exists

$$\lambda_2(L) = \min_{u \in X_1 \setminus \{0\}} \frac{\int_0^1 u''^2 dt}{\int_0^1 (L^2 q u'^2 + L^4 a u^2)\, dt}.$$

Note that $a_2(u,v)$ generates an equivalent norm in X_1 by Poincare-type inequalities and

$$\int_0^1 u'^2 dt = -\int_0^1 u u'' dt \le \frac{1}{2}\left(\int_0^1 u^2 dt + \int_0^1 u''^2 dt\right).$$

As before we can prove

Lemma 2.6. *There exists $L_2^* > 0$ such that $\lambda_2(L_2^*) = 1$*

$$(\lambda_2(L) - 1)(L - L_2^*) < 0, \quad L > 0.$$

Theorem 2.3. *There exists a number $L_2^* \in (L_2, L_1)$, where $L_j, j = 1, 2$ are given by* (3) *such that the problem* (P_1) *with $p < 0$ has only the trivial solution if $0 < L \le L_2^*$ and a nontrivial solution if $L > L_2^*$.*

3. Solvability of problem (P_2). In this section, we show that the problem (P_2) has a variational structure and its weak solutions are critical points of the functional

$$J(u;L) = \int_0^L \left(\frac{1}{2}\left(u'''^2 - Au''^2 + Bu'^2 - u^2\right) + \frac{1}{4} b(x) u^4\right) dx$$

in the space

$$Y(L) = \{H^3(0,L) : u(0) = u''(0) = u(L) = u''(L) = 0\}.$$

A function $u \in Y(L)$ is said to be a *weak solution* of (P_2) if

$$\int_0^L \left(u'''v''' - Au''v'' + Bu'v' - uv + bu^3 v\right) dx = 0, \quad \forall v \in Y.$$

The functional J is differentiable

$$\langle J'(u;L), v\rangle = \int_0^L \left(u'''v''' - Au''v'' + Bu'v' - uv + bu^3 v\right) dx,$$

and its critical points are weak solutions of (P_2). It is shown in [12] that a weak solution of (P_2) is also a classical solution.

Let us denote

$$L_2' := \pi \max\left(\left(B - \frac{A^2}{4}\right)^{1/2}, \left(1 - \frac{A^2}{4B}\right)^{1/6}\right) \tag{7}$$

and $L_1' := p_1^{1/2}$, where

$$\begin{aligned} p_1 &= q^{\frac{1}{3}} + \frac{1}{9} B^2 \pi^4 q^{-\frac{1}{3}} + \frac{1}{3} B \pi^2, \\ q &= \left(\frac{1}{2} + \frac{1}{27} B^3 + \frac{1}{18}\left(81 + 12B^3\right)^{1/2}\right)\pi^6. \end{aligned} \tag{8}$$

It is proved in [12] that problem (P_2) has only the trivial solution if $0 \leq L \leq L_2'$ and it has m distinct pairs nontrivial solution if $L > mL_1'$. The proof is divided in steps analogous as in Section 2 (cf. [12]).

Lemma 3.1. *The functional J is bounded from below on $Y(L)$, J has a bounded minimizing sequence and there exists a minimizer of J.*

Lemma 3.2. *Let $A^2 < 4B$ and $0 < L \leq L_2'$. Then (P_2) has only the trivial solution.*

Lemma 3.3. *The functional J satisfies the (PS) condition.*

Lemma 3.4. *Let $L > mL_1'$ for some $m \in \mathbf{N}$. Then there exists a set $K \subset Y(L)$ homeomorphic to $\mathbf{S}^{m-1}$ by an odd map, and $\sup_K J < 0$.*

Suppose that $A = 0, B > 0$ and make a change of variables $t = \frac{x}{L}$, $u(t) = u\left(\frac{x}{L}\right)$. The problem (P_2) is equivalent to the problem

$$(P_2') : \begin{cases} u^{iv} + L^4 B u'' + L^6 u - L^6 b u^3 = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u^{iv}(0) = u^{iv}(1) = 0. \end{cases}$$

and its weak solutions are critical points of the functional

$$J_1(u) = \frac{1}{2} \int_0^1 \left(u'''^2 + L^4 B u'^2 - L^6 u^2\right) dt + \frac{L^6}{4} \int_0^1 b u^4 dt$$

on the space $Y_1 = Y(1)$. The functional J_1 can be represented in the form

$$J_1(u) = a_3(u,u) - b_3(u,u) + \psi_3(u),$$

where

$$a_3(u,v) = \frac{1}{2} \int_0^1 \left(u''' v''' + L^4 B u' v'\right) dt$$

and

$$b_3(u,v) = \frac{1}{2} L^6 \int_0^1 uv dt, \ \psi_3(u) = \frac{L^6}{4} \int_0^1 b u^4 dt.$$

The forms a_3 and b_3 are symmetric positive bilinear forms, which satisfy (A_1) in Y_1, $\psi_3(u)$ is a positive convex functional and there exists

$$\lambda_3(L) = \min_{u \in X \backslash \{0\}} \frac{\int_0^1 \left(u'''^2 + L^4 B u'^2\right) dt}{L^6 \int_0^1 b u^2 dt}$$

As before we can prove

Lemma 3.5. *There exists $L_3^* > 0$ such that $\lambda_3(L_3^*) = 1$ and*

$$(\lambda_3(L) - 1)(L - L_3^*) < 0, \quad L > 0.$$

Theorem 3.1. *There exists a number $L_3^* \in (L_2', L_1')$, where $L_j', j = 1, 2$ are given by (7)–(8) such that the problem (P_2) with $A = 0, B > 0$ has only the trivial solution if $0 < L \leq L_3^*$ and a nontrivial solution if $L > L_3^*$.*

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НЕТРИВИАЛНИ РЕШЕНИЯ НА ГРАНИЧНИ ЗАДАЧИ ЗА ПОЛУЛИНЕЙНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ОТ ЧЕТВЪРТИ И ШЕСТ РЕД

Степан Агоп Терзиян, Мариа до Розарио Гросиньо

Доказани са резултати за съществуване на нетривиални решения на гранични задачи за полулинейни диференциални уравнения от четвърти и шести ред, които се срещат в теорията на фазовите преходи. Използувани са вариационни методи.